Working Paper Series 640
(ISSN 1211-3298)

Parameter Learning in Production Economies

Mykola Babiak
Roman Kozhan

CERGE-EI
Prague, April 2019

ISBN 978-80-7343-447-2 (Univerzita Karlova, Centrum pro ekonomický výzkum a doktorské studium)
ISBN 978-80-7344-500-3 (Národohospodářský ústav AV ČR, v. v. i.)

Parameter Learning in Production Economies ☆

Mykola Babiak [a,*] Roman Kozhan [b,**]

[a] CERGE-EI [b] Warwick Business School

April 2019

Abstract

We examine how parameter learning amplifies the impact of macroeconomic shocks on equity prices and quantities in a standard production economy where a representative agent has Epstein-Zin preferences. An investor observes technology shocks that follow a regime-switching process, but does not know the underlying model parameters governing the short-term and long-run perspectives of economic growth. We show that rational parameter learning endogenously generates long-run productivity and consumption risks that help explain a wide array of dynamic pricing phenomena. The asset pricing implications of subjective long-run risks crucially depend on the introduction of a procyclical dividend process consistent with the data.

Keywords: Parameter Learning, Equity Premium, Business Cycles, Markov Switching

JEL: D83, E13, E32, G12

☆We would like to thank Andrea Gamba, Michal Kejak, Ian Khrashchevskyi, Ctirad Slavik, Sergey Slobodyan and conference/seminar participants at the 2018 Lancaster-Warwick (LaWa) Workshop on Financial Econometrics and Asset Pricing, Warwick Business School, CERGE-EI and Università Ca' Foscari Venezia for their discussions and comments. We also thank the Centre for Scientific Computing at the University of Warwick for support with the supercomputing clusters. Mykola Babiak received financial support from the Charles University Grant Agency - GAUK (grant number 744218).

*CERGE-EI, a joint workplace of Charles University and the Economics Institute of the Czech Academy of Sciences, Politickych veznu 7, 111 21 Prague, Czech Republic.

**Warwick Business School, University of Warwick, Scarman Road, Coventry, CV4 7AL, UK.

Email addresses: mykola.babiak@cerge-ei.cz (Mykola Babiak), roman.kozhan@wbs.ac.uk (Roman Kozhan).

1. Introduction

Parameter learning has recently been proposed as an amplification mechanism for the pricing of macroeconomic shocks used to explain standard asset pricing moments. In the endowment economy, parameter uncertainty helps explain the observed equity premium, the high volatility of equity returns, the market price-dividend ratio and the equity Sharpe ratio (Collin-Dufresne, Johannes and Lochstoer, 2016; Johannes, Lochstoer and Mou, 2016). In contrast to the consumption-based approach, a production dynamic stochastic general equilibrium (DSGE) model endogenously generates consumption and dividends and, as a result, it becomes more challenging to explain asset pricing puzzles in a production-based setting while simultaneously matching the moments of macroeconomic fundamentals. In this paper, we study how the macroeconomic risks arising from parameter uncertainty improve the performance of a standard DSGE model in jointly reproducing salient features of the macroeconomic quantities and equity returns.

Kaltenbrunner and Lochstoer (2010) and Croce (2014) have argued that the presence of a small but persistent long-run risk component in the productivity growth process can endogenously generate long-run risks in consumption growth that help boost up moments of financial variables. However, these long-run risk components are difficult to identify in the data.[1] In contrast, we demonstrate that rational pricing of parameter uncertainty is a source of these subjective long-run risks in productivity growth. This suggests the importance of accounting for parameter uncertainty in the productivity growth process. It is not clear, however, if macroeconomic risks associated with rational learning about productivity growth amplify the moments of financial variables. If so, what is the magnitude of the effect? In this paper, we document a considerable amplification mechanism of rational parameter learning on asset prices.

We introduce parameter uncertainty in the technology growth process of an otherwise standard production-based asset pricing model. We depart from the extant macro-finance literature by presuming that the representative investor does not know the parameters of the technology process and learns about true parameter values from the data. In each period, he updates his beliefs in a Bayesian fashion upon observing newly arrived

[1] Croce (2014) empirically demonstrates the existence of such a predictable component; however, the results are not robust to estimation method and sample choice. Moreover, low values for goodness-of-fit statistics lead to a conclusion that there is considerable uncertainty about the model specification for productivity growth.

data. Rational learning about unknown parameters together with recursive preferences gives rise to subjective long-lasting macroeconomic risks. Coupled with endogenous long-run consumption risks due to consumption smoothing (Kaltenbrunner and Lochstoer, 2010) these risks are priced under the investor's preference for early resolution of uncertainty. The model generates higher equity Sharpe ratios, risk premia and volatility, as well as lower interest rates and price-dividend ratios relative to the standard framework. Additionally, the model with rational belief updating reproduces the excess return predictability pattern observed in the data. We further show that under certain calibrations of the elasticity of intertemporal substitution and a capital adjustment cost, parameter learning significantly magnifies propagation of shocks and hence helps to match the second moments and comovements of macroeconomic variables.

In our analysis, we restrict our attention to uncertainty about parameters governing the magnitude and persistence of productivity growth over the various phases of the business cycle. In particular, we examine the implications of learning about the transition probabilities and mean growth rates in a two-state Markov-switching process for productivity growth, where volatility of productivity growth is homoskedastic and known.[2] We consider two approaches to dealing with parameter uncertainty in the equilibrium models: anticipated utility (AU) and priced parameter uncertainty (PPU). The AU approach is common for most existing models, and assumes that economic agents learn about unknown parameters over time, but treat their current beliefs as true and fixed parameter values in the decision-making. For the PPU case, the representative investor calculates his utility and prices in the current period, assuming that posterior beliefs can be changed in the future. We quantify the impact of each type of parameter uncertainty pricing by comparing the results of AU and PPU with the full information (FI) model.

We begin our investigation by illustrating the economic importance of parameter uncertainty in the standard production economy with convex capital adjustment costs. The increased uncertainty due to unknown parameters in the productivity growth process creates a stronger precautionary saving motive, which leads to a lower risk-free rate. Fully

[2] There is a large strand of the literature emphasizing the importance of time-varying macroeconomic uncertainty (see, for example, Justiniano and Primiceri (2008); Bloom (2009); Fernandez-Villaverde, Guerron-Quintana, Rubio-Ramirez and Uribe (2011); Born and Pfeifer (2014); Christiano, Motto and Rostagno (2014); Gilchrist, Sim and Zakrajsek (2014); Liu and Miao (2014) and more recent studies by Leduc and Liu (2016); Basu and Bundick (2017); Bloom, Floetotto, Jaimovich, Saporta-Eksten and Terry (2018)). We leave the investigation of learning about volatility risks for future research.

rational learning about unknown parameters generates endogenous long-run risks in the economy, which in turn increase the mean and volatility of levered returns to the firm's payouts (Jermann, 1998). In contrast, fluctuations in parameter beliefs are not priced in the AU case. Thus, the PPU approach leads to around a two-fold increase in the risk premium (in addition to higher return volatility) on a levered firm's dividends, relative to the FI and AU cases. The combination of time-varying posterior beliefs and rational parameter learning is crucial for generating long-term predictability of excess returns by Tobin's Q, investment-capital, price-dividend and consumption-wealth ratios, as found in the empirical literature. The time-variation in beliefs leads to fluctuations in the equity risk premium and hence generates more predictability in the models with parameter uncertainty relative to the known parameter frameworks. Fully rational learning further magnifies the impact of belief revisions on the conditional equity premium and therefore there is more significant return predictability with PPU compared to AU. Specifically, the model with PPU closely replicates the increasing patterns (in absolute terms) of the regression coefficients and R^2's. In contrast, both the FI and AU models generate less predictability power and cannot match the magnitude of slope coefficients.

In terms of the macroeconomic variables, the benchmark model with parameter learning has a small effect on the unconditional second moments and a large impact on comovements of consumption, investment and output. In the sensitivity analysis, we further investigate how the impact of parameter learning on quantities changes for alternative calibrations of the inter-temporal elasticity of substitution and a capital adjustment cost. We find that a lower value of the inter-temporal elasticity of substitution, or a smaller capital adjustment cost, magnifies the effect of rational parameter learning on comovements between macroeconomic quantities. In particular, decreasing IES or adjustment costs for capital leads to significantly lower correlations between consumption, investment and output in the PPU model, while aggregate variables still remain highly correlated in the FI and AU economies. Thus, our evidence indicates that fully rational parameter learning generates additional macroeconomic risks, which interact with adjustment costs and elasticity of inter-temporal substitution, allowing us to better match the macro dynamics. These findings complement the results of Tallarini (2000), Campanale, Castro and Clementi (2010) and Liu and Miao (2015), who find no effect of increasing agents' sensitivity to risk on the macroeconomic variables.

There are however several issues that this version of the model with parameter learning does not resolve. Although parameter learning increases the equity premium, the magnitudes are still too small compared to the historically observed statistics. The main reason for this underperformance of the model is found in countercyclical dynamics of a firm's endogenous dividends in the production economy. Therefore, we further consider pricing a claim to exogenous market dividends that are directly calibrated to reconcile dividend dynamics. In this way, we are able to verify that the low equity premium arises not because of an insufficient amplification effect of parameter uncertainty, but due to the inability of the production economy to generate procyclical dividends.

When pricing a claim to calibrated dividends, we find that the PPU model with a century-long prior learning period and unbiased prior beliefs generates an average equity premium, equity volatility, equity Sharpe ratio, risk-free rate, and a level and autocorrelation of the price-dividend ratio close to the values observed in the data. Learning provides a significant improvement in the performance of the production model relative to the FI and AU cases, which cannot match these standard asset pricing moments. Furthermore, learning generates long-lasting effects on asset prices as the size of the risk premium and its volatility remain high even after 200 years of a prior learning period. To better understand the source of the model's improvement, we look at the conditional dynamics of the key asset prices and conditional moments. We find that parameter learning generates a much stronger amplification mechanism in bad times than in good, generating countercyclical fluctuations in the conditional risk premium, volatility and Sharpe ratios that are consistent with the data.

In sum, fully rational pricing of parameter uncertainty improves the fit of the standard production economy to a large array of empirical regularities, though parameter learning alone cannot fix a common problem of countercyclical firm dividends in the frictionless economy. In order to maintain a desired feature of endogenous dividends, we consider an extension of the benchmark model that would generate a more procyclical firms' levered payouts consistent with the data. In particular, the extension incorporates the idea of costly reversibility (see, for example, Abel and Eberly (1994, 1996), Hall (2001) and Zhang (2005) among others), which means that firms face higher costs in cutting than in expanding capital stocks. Intuitively, the mechanism of investment frictions works as follows. In bad times, it is more difficult for a representative firm to reduce investment, due to higher costs that would lead to a smaller drop in investment compared to the symmetric capital adjustment cost. Thus, net profits after deducting investment appear less countercyclical. With the financial leverage, a firm's dividends are the sum of a firm's profits and the net balance of the long-term debt. The latter is proportional to capital and therefore declines in the recession. The overall sum of the profits and net issuance of the long-term debt results in procyclical dividends.

We find that the unconditional statistics of the levered returns to endogenous firm dividends are now much closer to the data. In particular, the extended model accounts for a large equity premium, around two thirds of equity volatility, and furthermore, it matches well to the mean, volatility and autocorrelation of the price-dividend ratio. The quadratic asymmetric adjustment cost function further lowers the correlations between macroeconomic variables. The results of the benchmark calibration and the extended model confirm our findings that for all relevant moments parameter learning provides a substantial improvement relative to the FI and AU cases.

The main mechanism of this paper is closely related to the work of Collin-Dufresne, Johannes and Lochstoer (2016) who study a similar learning problem in the endowment economy. Our analysis differs from theirs in the following ways. First, we extend their methodology to a production economy setting and explore joint implications of parameter uncertainty for macroeconomic quantities and asset prices. Second, relative to the endowment model, one needs to generate procyclical dividends in the production economy to obtain a significant amplification of equity moments by parameter learning. We document this result by pricing a claim to exogenous calibrated dividends. We further confirm this finding in the extension of the model with costly reversibility, which generates endogenous procyclical firm's payoffs. Third, rather than exploring the impact of learning in a rare events model (Rietz, 1988; Barro, 2006), we instead estimate the production parameters by the expectation maximization algorithm from the postwar U.S. data. Even though the estimated process for productivity growth does not reflect rare states that are naturally difficult to be learned about due to their rareness, fully rational parameter learning still matches well financial moments in our setting with more frequent states. The main reason for this is that long-run consumption risks generated by consumption smoothing (see Kaltenbrunner and Lochstoer, 2010) magnify the impact of endogenous long-run productivity risks originating from belief revisions on asset prices; therefore, less is needed in terms of the speed of parameter learning.

Our paper also speaks to macro-finance research in the production-based economies. Cagetti, Hansen, Sargent and Williams (2002) is one of the first examples of a business cycle model with parameter learning. In their paper, Cagetti, Hansen, Sargent and Williams (2002) consider a signal extraction problem about the unobservable mean growth rate of technology shocks. However, they do not study the implications of incomplete information for quantities and asset prices, a key focus of our analysis. In a recent paper, Jahan-Parvar and Liu (2014) examine a production economy with learning about a latent state in a productivity growth process following a two-state hidden Markov chain. Their paper is an adaption of the endowment economy with ambiguity preferences (Ju and Miao, 2012) to a production setting. The key differentiators of our study from Jahan-Parvar and Liu (2014), as well as the extant literature on learning in a business cycle model, is a multidimensional learning problem and rational pricing of parameter beliefs.

Our paper is also related to the long-run risks models introduced by Bansal and Yaron (2004). Kaltenbrunner and Lochstoer (2010) and Croce (2014) investigate the original source and implications of long-run productivity and consumption risks. In relation to these studies, we do not explicitly incorporate long-run dynamics in productivity growth by adopting the model of Bansal and Yaron (2004). In our paper, the subjective long-run macroeconomic risks appear as a result of Bayesian learning about true parameter values. Our approach is complementary to the existing long-run risks literature and in fact provides the empirical investigation of possible origins of long-run productivity risks.

The paper proceeds as follows. Section 2 presents the formal model, Section 3 investigates the quantitative implications of parameter learning for quantities and asset prices. Section 4 performs sensitivity analysis. Section 5 concludes.

2. The Model

In this section, we present a production-based asset pricing model (Jermann, 1998; Campanale, Castro and Clementi, 2010; Croce, 2014; Kaltenbrunner and Lochstoer, 2010). The model is a standard business cycle framework (Kydland and Prescott, 1982; Long and Plosser, 1983) populated by a representative firm with Cobb-Douglas production technology and capital adjustment costs, and a representative household with

Epstein-Zin preferences. The firm produces a single consumption-investment good using labor and capital as inputs subject to productivity shocks. The household participates in the production process by working for the firm and investing in capital. Additionally, the representative investor trades firm shares and risk-free bonds to maximize lifetime utility of a consumption stream subject to a sequential budget constraint. Ultimately, the representative firm maximizes its value by choosing labor and investment demand. Our objective is to investigate the impact of learning about parameters in the productivity process on the moments of macroeconomic quantities and equity returns.

2.1. The Representative Household

We assume that a representative household has the recursive utility of Epstein and Zin (1989):

$$U_t = \left\{ (1-\beta) C_t^{1-1/\psi} + \beta \left(E_t \left[U_{t+1}^{1-\gamma} \right] \right)^{\frac{1-1/\psi}{1-\gamma}} \right\}^{\frac{1}{1-1/\psi}} \tag{1}$$

where U_t denotes the household's continuation utility, C_t denotes aggregate consumption, E_t denotes the expectation operator, $\beta \in (0,1)$ is the discount factor, $\psi > 0$ represents the elasticity of inter-temporal elasticity of substitution (EIS), $\gamma > 0$ represents the risk aversion parameter. For simplicity, we will assume that the household inelastically supplies one unit of labor; thus, the household's intra-period utility depends only on consumption.

It is straightforward to derive the stochastic discount factor:

$$M_{t+1} = \beta \left(\frac{C_{t+1}}{C_t} \right)^{-1/\psi} \left(\frac{U_{t+1}}{\left(E_t \left[U_{t+1}^{1-\gamma} \right] \right)^{\frac{1}{1-\gamma}}} \right)^{1/\psi - \gamma} \tag{2}$$

The key feature is a separation of agent's relative risk aversion from the elasticity of inter-temporal substitution. If $\gamma \neq \frac{1}{\psi}$, the utility function is not time-additive, and the stochastic discount factor has two components. The first term represents the kernel of the power utility, while the second term is the adjustment of the Epstein-Zin utility. In this paper, we set $\gamma > \frac{1}{\psi}$ and, thus, the household prefers earlier resolution of uncertainty. When the household's continuation utility U_{t+1} is below the certainty equivalent of this continuation utility, the second ingredient in the pricing kernel increases, raising a premium for long-run risks.

We also aim to investigate the impact of learning about unknown parameters governing the technology process. Although we do not introduce long-run productivity risks directly by assuming a persistent component in productivity growth (Croce, 2014; Kaltenbrunner and Lochstoer, 2010), Bayesian belief updating will generate subjective long-run productivity risks. Since the representative household has a preference for early resolution of uncertainty and is particularly averse to such long-run risks, parameter learning will generate quantitatively significant macroeconomic risks, improving the performance of the model in explaining salient features of the data.

2.2. The Representative Firm

The representative firm produces the consumption good using a constant returns to scale Cobb-Douglas production function:

$$Y_t = K_t^{\alpha}(A_t N_t)^{1-\alpha}, \tag{3}$$

where Y_t is the output, K_t is the capital stock, N_t is labor hours, and A_t is an exogenous, labor-enhancing technology level (which we also refer to as productivity). For simplicity, we assume that the representative household supplies the fixed amount of labor hours, which are exogenously set $N_t = 1$.

The firm's capital accumulation equation incorporates capital adjustment costs and is formally defined by:

$$K_{t+1} = (1-\delta)K_t + \varphi(I_t/K_t)K_t,$$

where $\delta \in (0,1)$ is the capital depreciation rate, $I_t = Y_t - C_t$ denotes gross investment, and $\varphi(\cdot)$ is the capital adjustment cost function given by:

$$\varphi(x) = a_1 + \frac{a_2}{1-1/\xi}x^{1-1/\xi}, \tag{4}$$

where ξ is the elasticity of the investment rate to Tobin's q. We follow Boldrin, Christiano and Fisher (2001) and choose the constants a_1 and a_2 such that there are no adjustment costs in the non-stochastic steady state.[1]

2.3. Technology

We consider a parsimonious two-state Markov switching model for the productivity growth rate $\Delta a_t = \ln\left(\frac{A_t}{A_{t-1}}\right)$:

$$\Delta a_t = \mu_{s_t} + \sigma\varepsilon_t,$$

[1]Specifically, $a_1 = \frac{1}{\xi-1}\left(1-\delta-\exp(\bar{\mu})\right)$, $a_2 = \left(\exp(\bar{\mu})-1+\delta\right)$, where $\bar{\mu}$ is the unconditional mean of μ_{s_t}. We find steady state values of the remaining quantities from the conditions $\varphi\left(\frac{I}{K}\right)=1$, $\varphi^{'}\left(\frac{I}{K}\right)=1$. In particular, the steady state investment-capital ratio is $\frac{I}{K} = \exp(\bar{\mu})-1+\delta$.

where $\varepsilon_t \overset{\text{iid}}{\sim} N(0,1)$, s_t is a two state Markov chain with transition matrix:

$$\Pi = \begin{bmatrix} \pi_{11} & 1-\pi_{11} \\ 1-\pi_{22} & \pi_{22} \end{bmatrix},$$

where $\pi_{ii} \in (0,1)$. We label $s_t = 1$ the "good" regime with high productivity growth and $s_t = 2$ the "bad" regime with low productivity growth.

2.4. Equilibrium Asset Prices

In the competitive equilibrium of the economy, the representative household works for the firm and trades its shares to maximize the lifetime utility over a consumption stream. The representative firm chooses labor and capital inputs (through investment) to maximizes the firm's value, the present value of its future cash flows. The firm's maximization problem implies the following equilibrium conditions for gross return $R_{j,t+1}$ of the asset j between period t and $t+1$:

$$E_t\left[M_{t+1}R_{j,t+1}\right] = 1. \tag{5}$$

In particular, the equation above is satisfied by the investment return, $R_{I,t+1}$, defined by:

$$R_{I,t+1} = \frac{1}{Q_t}\left[Q_{t+1}\left(1-\delta+\varphi\left(\frac{I_{t+1}}{K_{t+1}}\right)\right) + \frac{\alpha Y_{t+1} - I_{t+1}}{K_{t+1}}\right], \tag{6}$$

where Q_t is Tobin's marginal Q :

$$Q_t = \frac{1}{\varphi'\left(\frac{I_t}{K_t}\right)} = \frac{1}{a_2}\left(\frac{I_t}{K_t}\right)^{1/\xi}.$$

The return on investment can be interpreted as the return of an equity claim to the unlevered firm's payouts (Restoy and Rockinger, 1994). As the firm behaves competitively, the labor input is chosen at a level equal to its marginal product: $w_t = \partial Y_t/\partial N_t = (1-\alpha)A_t^{1-\alpha}K_t^{\alpha}N_t^{-\alpha} = (1-\alpha)Y_t/N_t$. The unlevered firm value, FV_t, is given by $FV_t = Q_tK_{t+1}$, and the firm's unlevered dividends, D_t, are defined by:

$$D_t = Y_t - w_tN_t - I_t = \alpha Y_t - I_t. \tag{7}$$

Since the observed aggregate stock market dividends are not directly comparable to the endogenous payouts defined above,[2] we consider pricing levered equity claims.

[2] As noted by other studies, unlevered cash flows and investment returns are not directly observed in reality. Additionally, the equity prices observed on the market are for leveraged corporations, in contrast to unlevered dividend payments of production companies in the model.

We introduce financial leverage in the spirit of Jermann (1998) by presuming that in each period the firm issues long-term bonds for a fixed fraction of capital and pays the outstanding debt from previous periods. Note that Modigliani and Miller conditions hold in our model and, thus, introducing the financial leverage does not change the equilibrium allocations. It only influences the dynamics of a firm's payouts and the way we report the returns on a claim to the firm's dividends. In particular, the financial leverage increases volatility of dividends and makes equity returns more risky.

Following Jermann (1998), we assume that the firm issues n period discount bonds and pays back its outstanding debt of n period maturity in each period. The fraction ω of the firm's capital K_t at time t is invested in long-term bonds. Denoting the price of the n period discount bonds at time t by $B_{t,n}$ the dividends stream is given by:

$$D_t^l = Y_t - w_t N_t - I_t + \omega K_t - \omega K_{t-n}/B_{t-n,n}, \tag{8}$$

where the first part, $Y_t - w_t N_t - I_t$, represents the operating cash flow of an unlevered claim, whereas the second part, $\omega K_t - \omega K_{t-n}/B_{t-n,n}$, is the difference between proceeds from newly issued bonds in period t at the price $B_{t,n}$ and repayments of the bonds purchased in period $t-n$ at the price $B_{t-n,n}$. We assume that the Modigliani and Miller Theorem holds in this setting. This implies that the financial policy above does not affect a firm's value and investment decision.

The price of the n-period bonds is defined recursively by:

$$B_{t,n} = E_t\left[M_{t+1} B_{t+1,n-1}\right], \tag{9}$$

with the boundary condition $B_{t,0} = 1$ for any t. We denote the price of the (levered) equity claim by P_t^l, and the (levered) equity return by $R_{t+1}^l = (P_{t+1}^l + D_{t+1}^l)/P_t^l$. By (5) and (8), the equity price satisfies $P_t^l = E_t(M_{t+1}(D_{t+1}^l + P_{t+1}^l))$ and can be readily computed by the formula $P_t^l = FV_t - DV_t$, where FV_t represents a firm's value, DV_t denotes debt value of all outstanding bonds from period $t-n+1$ to period t. Specifically:

$$DV_t = \sum_{j=1}^{n} \frac{B_{t,j}\,\omega K_{t-n+j}}{B_{t-n+j,n}}.$$

3. Results

We start with calibrating a benchmark model and analyzing the implications of parameter uncertainty in the productivity growth process for macroeconomic quantities and asset returns. We focus our attention on the stylized facts observed in the U.S. post-World War II data. Specifically, we compare the model-generated statistics with the historical data for 1952:Q1-2016:Q4. Macroeconomic data on consumption, investment, capital, and output are taken from the U.S. National Income and Product Accounts (NIPA) as provided by the Bureau of Economic Analysis (BEA). The asset returns data and dividends are from the Center for Research in Security Prices (CRSP). The model is calibrated at a quarterly frequency.

Since the model does not admit an analytical solution, we solve for equilibrium allocations numerically through value function iteration. We extend the Collin-Dufresne, Johannes and Lochstoer (2016) solution methodology to the production-based setting. The detailed description of the numerical methods is presented in the Appendix. Having solved the model, we generate 1,000 simulations of the economy with the sample length of 260 periods and report statistics of asset returns and macroeconomic quantities corresponding to their empirical counterparts.

3.1. Parameter Values

Panel A in Table 1 reports the parameter values of an investor's preferences, production and capital adjustment cost functions. We choose these parameter values similarly to the existing real-business cycle models. In particular, the constant capital share in a Cobb-Douglas production function (α) is 0.36, and the quarterly depreciation rate (δ) is 0.02. We set the capital adjustment cost parameter (ξ) equal to 4, which yields volatility of investment growth and consumption growth relatively close to the data. We further choose the constants (a_1, a_2) in the capital adjustment cost function such that there are no adjustment costs in the non-stochastic steady state.

The preference parameters are also consistent with the macroeconomic literature. The coefficient of relative risk aversion (γ) is equal to 10, the upper bound of an interval considered plausible by Mehra and Prescott (1985). The subjective discount factor (β) is set to 0.9945. This value allows the benchmark calibration to generate the low unconditional risk-free rate. There is no consensus in the literature about the value of the elasticity of inter-temporal substitution. We follow the disaster risk literature (Gourio, 2012) and long-run risks models (Bansal and Yaron, 2004; Ai, Croce and Li, 2013; Bansal, Kiku, Shaliastovich and Yaron, 2014) by setting EIS (ψ) to 2.

Following the methodology of Stock and Watson (1999), we use the macroeconomic

Table 1
Benchmark Calibration

Parameter	Description	Value
Panel A: Preferences, Production and Capital Adjustment Costs Functions, and Financial Leverage		
β	Discount factor	0.9945
γ	Risk aversion	10
ψ	EIS	2
α	Capital share	0.36
δ	Depreciation rate	0.02
ξ	Adjustment costs parameter	4
a_1	Normalization	-0.0075
a_2	Normalization	0.3877
Panel B: Markov-switching Model of Productivity Growth		
π_{11}	Transition probability from expansion to expansion	0.947
π_{22}	Transition probability from recession to recession	0.662
μ_1	Productivity growth in expansion	0.54
μ_2	Productivity growth in recession	-1.53
σ	Productivity volatility	1.36

This table reports the parameter values in the benchmark calibration. Panel A presents preferences parameters, values in the production and adjustment costs functions. Panel B shows the maximum likelihood estimates of parameters in a two-state Markov-switching model for productivity growth. We obtain these estimates by applying the expectation maximization algorithm (Hamilton, 1990) to quarterly total factor productivity growth rates from 1952:Q1 to 2016:Q4.

data to construct the cumulative Solow residuals. We further scale these residuals by the labor share $(1 - \alpha)$ in order to interpret them as labor-augmenting technology. We estimate a two-state Markov switching process of quarterly productivity growth rates by applying the expectation maximization algorithm developed by Hamilton (1990). Panel B in Table 1 reports the maximum likelihood estimates for the transition probabilities (π_{ii}), productivity growth rates (μ_i) as well as the constant volatility (σ). Productivity is estimated to grow at the quarterly rate of about 0.54 percent in expansions and about -1.53 percent in recessions. The productivity volatility comes out around 1.36 percent. The transition probability to the expansion (recession) conditional on being in the expansion (recession) is estimated around 0.947 (0.662). These numbers imply the average duration of the high-growth expansion state of about 18.87 quarters and the average duration of the low-growth recession of about 2.96 quarters. Our maximum likelihood estimates are broadly consistent with the values reported by Hamilton (1989) and Cagetti, Hansen, Sargent and Williams (2002).

Once we solve the model according to the calibration above, we introduce financial leverage by assuming the representative firm issues long-term bonds with a maturity of fifteen years. Because the Modigliani-Miller theorem holds, this only changes the levered returns and dividends but does not influence the equilibrium allocation of the economy. With financial leverage, equity value depends on the market value of the firm and the total debt oustanding bonds. All valuations endogenously depend on the equilibrium investment decision. For each model, we calibrate (ω) in order to match the average debt-to-equity ratio of around 1:1. Therefore, the leverage parameter across different models is in the interval $[1\%, 1.1\%]$.

3.2. Parameter Uncertainty

In this paper, we consider five parameters in the productivity growth process. We employ conjugate priors for each unknown parameter in order to obtain conjugate posteriors via Bayesian updating. If all parameters are assumed to be unknown for the agent, we obtain a 10-dimensional vector of state variables including the current regime of the Markov chain, capital stock, time and hyperparameters of prior distributions. In addition to the curse of dimensionality, the numerical solution methodology in the production-based setting requires the solution of the agent's maximization problem for every combination of state variables in each period. This makes the model solution especially slow. To mitigate complexity in the model solution, we investigate the impact of uncertainty about the transition probabilities and mean growth rates, whereas a volatility parameter is assumed to be known.[3] Furthermore, our analysis assumes homoskedastic volatility of productivity growth, though a large strand of the macroeconomic literature documents the importance of time-varying uncertainty on macroeconomic variables and asset returns. We leave the important investigation of the implications of learning about volatility risk and regime switches in volatility of productivity growth for future research.

Having decided which parameters are unknown for the agent in the production economy, we consider the two approaches to dealing with parameter uncertainty: priced parameter uncertainty and anticipated utility. PPU implies the economic agents learn about unknown parameters from the data and rationally take into account the changing beliefs while making their decisions. AU assumes the decision-makers learn about unknown parameters over time but in each period of time they treat their current beliefs as "true" values. Thus, AU agents ignore the possibility that parameters might actually change

[3] We motivate our choice of unknown parameters by the results in the consumption-based asset pricing model. Specifically, Collin-Dufresne, Johannes and Lochstoer (2016) conclude that uncertainty about variance has a negligible effect on asset prices. Meanwhile, learning about transition probabilities has long-lasting asset pricing implications. Mean growth rates are harder to learn than volatilities, though the implications are less pronounced compared to learning about unknown transition probabilities.

in the future. Since the posterior beliefs are martingales, Bayesian learning generates subjective long-run risks in the economy, which would be priced under rational beliefs pricing, unlike the AU case. To evaluate the impact of these additional macroeconomic risks, we consider different specifications of the production economy for the comparison analysis. We start by solving three frameworks: our preferred benchmark with full information about parameters in the productivity growth process and an identical calibration with unknown parameters incorporating either PPU or AU. This comparison allows us to isolate the impact of rational pricing of parameter uncertainty. Furthermore, we study the role of an investor's prior knowledge by injecting training samples with different lengths into the model. By experimenting with the prior samples, we can evaluate how persistent the impact of rational beliefs updating is.

In sum, we use standard, conjugate priors distributions for the unknown parameters: beta and normal distributions for the transition probabilities and mean growth rates, respectively. We choose the hyperparameters of the distributions such that initial beliefs are centered at the true values of uncertain parameters estimated from the postwar sample. Furthermore, we solve the models with parameter uncertainty based on the prior training samples of 100, 150 or 200 years of initial learning. Thus, although calibrating initial beliefs based on the historical data may be a realistic feature and will certainly improve the model's performance due to pessimism induced by the Great Depression and both World Wars, our results do not require pessimistic prior specifications and are based on the information contained in the postwar data. For each specification, we numerically solve the production economy using the methodology outlined in the Appendix.

3.3. Pricing a Claim to Firm's Levered Dividends

In this section, we quantitatively analyze the impact of uncertainty about the transition probabilities (π_{11}, π_{22}) and mean growth rates (μ_1, μ_2) in the production economy of this paper.[4] The macroeconomic variables of our interest are consumption, investment and output. The financial variables include short-term (one quarter) and long-term (15 years) risk-less bonds and equity claim on the firm's leveraged dividends. First, we assess the implications of parameter uncertainty in the production economy by comparing

[4]The Appendix provides extensive details of the numerical solution methodology in different settings. The model-generated statistics with uncertainty about the transition probabilities are similar to the results presented in the main text with both unknown probabilities and mean growth rates. For brevity, these results are not reported but are available upon request.

model-implied unconditional moments of quantities and asset returns to their sample counterparts. Second, we study the impulse responses of quantities to a regime switch in the productivity growth. Finally, we check the ability of the economy to reproduce the long-horizon predictability of excess equity returns.

Unconditional Moments. Panel A in Table 2 presents business cycle moments of macroeconomic variables from simulations of models as well as the U.S. post WWII statistics. The data column shows that output is more volatile than consumption but less volatile than investment. Also, there is a significant correlation between the three series, especially between investment and output growth. Comparing the empirical moments with the model-generated statistics, all three models with PPU, AU and fixed parameters explain the empirical moments reasonably well. Relative to the case of known parameters and AU, rational pricing of beliefs with 100 years of prior learning slightly increases investment growth volatility, lowers consumption growth volatility and brings the correlations between the macro quantities closer to the data. However, parameter learning has quantitatively marginal effects on the macro dynamics.

In contrast, Panel B in Table 2 shows that priced parameter uncertainty improves more significantly the performance of the real business cycle model in terms of financial moments. The last two columns in Table 2 show that the production economy with known parameters, or with unknown parameters but AU pricing (for the AU case, we report the results only with a prior period of 100 years), generates a too high average risk-free rate and price-dividend ratio as well as a too low mean and volatility of excess equity returns compared to the data. Columns 3 to 6 shows that rationally taking into account parameter uncertainty in the productivity growth process leads to a lower risk-free rate and price-dividend ratio. The risk premium is almost two times higher with parameter learning, equity volatility and the price of risk also increase in this case.

Although the financial moments are amplified in the model with priced parameter uncertainty, they are still too small compared to the data. The reason for a very small equity premium and equity volatility in the production economy is the countercyclical dynamics of dividends growth as documented by Kaltenbrunner and Lochstoer (2010) among others. This is in contrast to the observed procyclical dividends in the data. Panel B in Table 2 indicates that a firm's payouts are strongly negatively correlated with consumption growth in all models, while we document that the corresponding correlation

Table 2
Sample Moments

	Data	PPU				AU	FI
		100 yrs	150 yrs	200 yrs	∞ yrs		
Panel A: Macroeconomic Quantities							
$\sigma(\Delta c)$	1.26	1.30	1.32	1.32	1.33	1.31	1.33
$\sigma(\Delta i)$	4.51	3.51	3.46	3.45	3.47	3.47	3.44
$\sigma(\Delta y)$	2.41	1.97	1.96	1.96	1.96	1.94	1.94
$ar1(\Delta c)$	0.32	0.14	0.16	0.17	0.19	0.19	0.18
$\rho(\Delta i, \Delta y)$	0.72	0.97	0.98	0.99	0.99	0.99	0.99
$\rho(\Delta c, \Delta y)$	0.52	0.96	0.97	0.98	0.99	0.98	0.98
$\rho(\Delta c, \Delta i)$	0.36	0.90	0.92	0.94	0.96	0.95	0.95
Panel B: Financial Variables							
$E(R_f) - 1$	1.44	1.64	1.78	1.87	2.01	2.13	2.14
$\sigma(R_f)$	1.07	0.41	0.38	0.37	0.34	0.31	0.31
$\sigma(M)/E(M)$		0.29	0.27	0.25	0.23	0.19	0.19
$E(\Delta d^l)$	2.06	0.65	0.77	0.84	0.95	1.07	1.08
$\sigma(\Delta d^l)$	10.38	13.35	13.74	14.20	15.77	17.09	16.19
$ar1(\Delta d^l)$	0.25	−0.01	−0.01	−0.01	−0.01	−0.01	−0.02
$\rho(\Delta c, \Delta d^l)$	0.44	−0.53	−0.58	−0.61	−0.63	−0.62	−0.60
$E(R^l - R_f)$	5.51	2.92	2.60	2.25	1.84	1.56	1.52
$\sigma(R^l - R_f)$	16.55	5.41	5.25	4.83	4.49	4.41	4.38
$E(p^l - d^l)$	3.19	3.38	3.45	3.55	3.65	3.67	3.67
$\sigma(p^l - d^l)$	0.33	0.31	0.31	0.32	0.35	0.36	0.36
$ar1(p^l - d^l)$	0.97	0.95	0.95	0.95	0.95	0.95	0.95

This table reports the average moments from 1,000 simulations of 260 quarters of the data from the production economy considered in this paper, where the transition probabilities and mean growth rates are assumed to be unknown. The historical data moments are reported in the data column and correspond to the U.S. data from 1952:Q1 to 2016:Q4. The PPU column refers to the production economy with rational pricing of parameter uncertainty, whereas the AU column refers to the production economy with AU pricing. In both cases, parameter uncertainty includes unknown transition probabilities and mean growth rates. The FI column presents the results of the full information case where the parameters are known. $E(x)$ and $\sigma(x)$ denote the average sample mean and standard deviations of x, respectively. $ar1(x)$ and $\rho(x, y)$ denote the average sample autocorrelation of x and correlation between x and y, respectively. All statistics are expressed in annualized terms, except for market price of risk given in percent, whereas correlations and autocorrelations are expressed in quarterly terms.

between consumption and dividend growth is about 0.44 in the data. A number of studies (Uhlig, 2007; Belo, Lin and Bazdresch, 2014; Favilukis and Lin, 2016) introduce wage rigidity in the standard production model in order to generate more volatile and procyclical dividends. This extension of the model can further improve our results and possibly magnify the effect of parameter learning. However, we leave the investigation of the interplay between sticky wages and parameter uncertainty for future research. In this paper, we will directly calibrate the firm's dividends process to the empirical counterpart.

Impulse Response Functions. Figure 1 illustrates the response of the economy with unknown transition probabilities to a typical recession lasting for 1 quarter, 3 quarters and 2 years. The economy is assumed to grow at the mean growth μ_1 and μ_2 in each state. Before the economy enters the recession, the representative investor holds unbiased beliefs about the uncertain parameters (the transition probabilities π_{11} and π_{22}) assuming a 100-year prior period. We feed these simulated paths of beliefs and productivity growth series into the model and calculate the equilibrium quantities as described in Appendix C.

The top panels in Figure 1 show the mean beliefs about the transition probabilities. Upon the onset of the recession, the mean belief about staying in the good regime falls sharply and stays at the same level during the recession. Once the economy returns back to the high growth state, the investor gradually updates his beliefs about π_{11} upward. In contrast, learning about π_{22} happens only in the recession. The random durations of 1 quarter, 3 quarters and 2 years correspond to the realization of a short economic decline, an average recession and a long downturn, respectively. When the agent experiences an average duration of the recession, his belief about π_{22} increases but then returns back to the initial value. The mean belief about π_{22} remains permanently lower (higher) relative to the initial belief in the case of the recession that is shorter (longer) than the average downturn.

The middle panels of Figure 1 present the impulse responses of macroeconomic quantities and equity prices. Given that productivity growth declines and the investor's probability beliefs about π_{11} drop, the capital stock declines upon the bad news in the economy and consequently leads to a reduction in investment and consumption. As productivity stays low and probability beliefs become more pessimistic, macroeconomic variables continue to fall and start to recover only after the economy exits the recession. The stock prices fall in response to switching to the low productivity growth regime. Also, the realized equity returns are smaller in the recession due to low productivity growth and bounce back to the original rate in the expansion. Since consumption dynamics predicts the high marginal utility when productivity growth is low, equity returns are positively exposed to the regime switching in mean productivity growth. Overall, the model predicts the dynamics of consumption, investment, equity prices and equity returns consistent with the data.

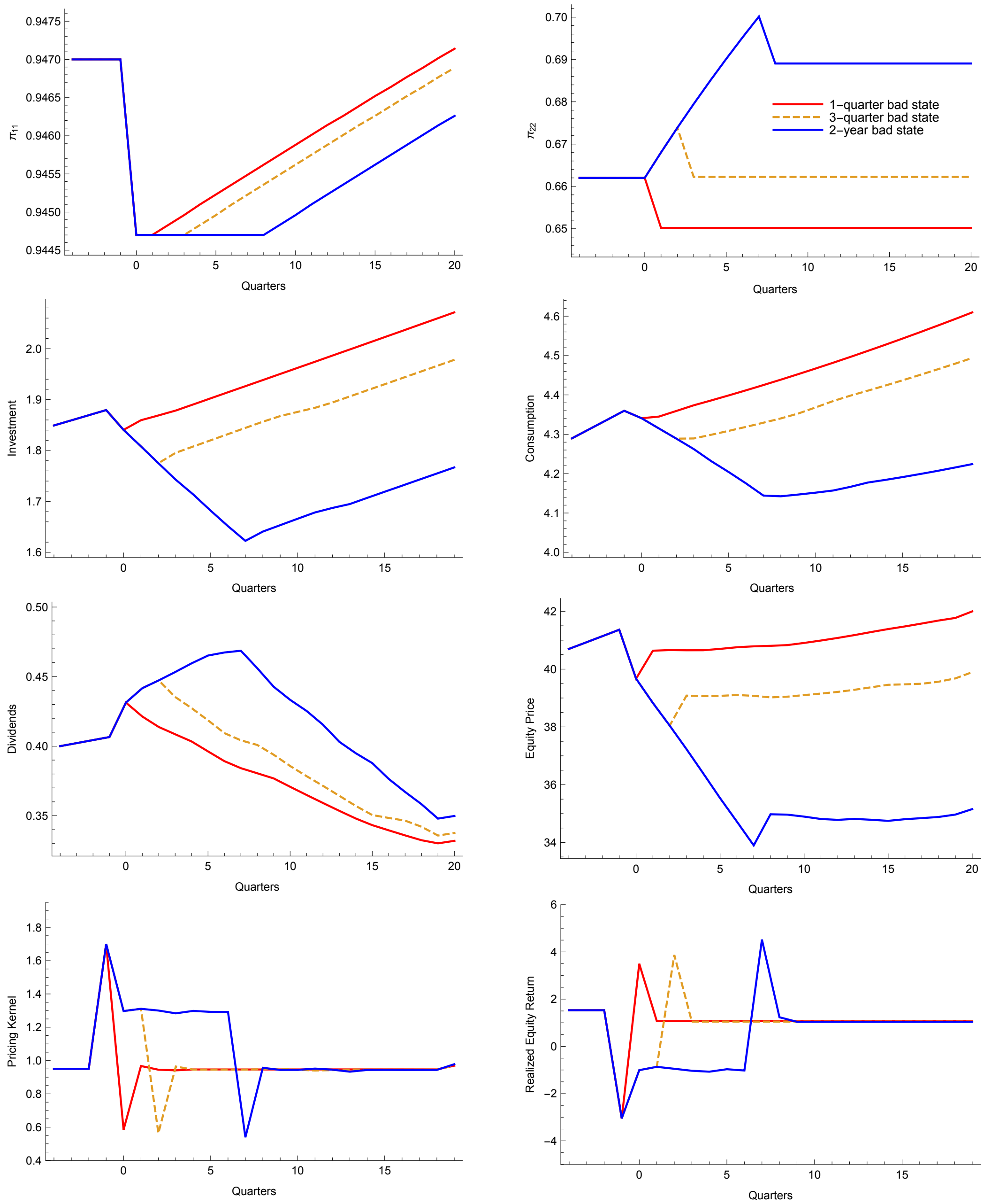

Figure 1: Beliefs, Impulse Responses and Recession Realizations. The figure shows the dynamics of investor's beliefs and the impulse responses of macroeconomic and financial variables to a typical recession in the model with unknown transition probabilities. Initially, the economy stays in the expansion for a long period, and the investor holds unbiased mean beliefs about the transition probabilities (π_{11} and π_{22}) based on a 100-year prior period. The panels show three simulated paths with the duration of the recession state equal to 1 quarter, 3 quarters and 2 years.

Turning to endogenous firm's payouts, the dividends increase following the recession realization, in contrast to a procyclical movement observed in the data. The unlevered dividends, which are not reported in Figure 1, are approximately equal to profits minus investment. Since profits in the model are smooth relative to investment and investment is procyclical, the endogenous unlevered firm's dividends are strongly countercyclical and would initially increase on impact and then grow at the original rate. Similarly, the levered firm's dividends reported in Figure 1 increase upon entering the recession but start to decline when the economy returns to the growth state. The reason is that a sharp decline in the mean beliefs about π_{11} has a negative and long-lasting impact on the capital stock. Since the agent invests a constant proportion of the firm's capital in the long-term bonds, the current profits from trading the long-term bonds remain negative until capital recovers to the initial level.

Return Predictability. A large strand of the empirical literature documents that excess returns at an aggregate level can be predicted by variables like the investment-capital ratio (Cochrane, 1991; Bansal and Yaron, 2004), Tobin's Q (Pontiff and Schall, 1998; Lewellen, 2004), the dividend-price ratio (Campbell and Shiller, 1988; Fama and French, 1989) and the consumption-wealth ratio (Lettau and Ludvigson, 2001). In this section, we compare the long-term predictability patterns generated by the production economy with parameter uncertainty (both the PPU and AU cases) and fixed parameters to the predictability observed in the post-war data. The conclusion of the extensive empirical literature is that high dividend yields, high book-to-market and consumption-wealth ratios predict high future excess returns, whereas high investment rates forecast low future excess returns. Furthermore, the predictive regressions suggest that the slope coefficients (in absolute terms) and R^2's are relatively large and tend to increase over the forecast horizon. These regularities pose a significant challenge for the standard real business cycle model.

Tobin's Q, the investment-capital and consumption-wealth ratios are endogenously specified in our production economy. Furthermore, we follow Epstein and Zin (1989) and calculate the wealth-consumption ratio as:

$$\frac{W_t}{C_t} = \frac{1}{1-\beta}\left(\frac{U_t}{C_t}\right)^{1-1/\psi},$$

where the equilibrium allocations of the agent's utility and consumption are endogenously determined. Using these model-generated quantities, we run the abovementioned predictive regressions and report results in Table 3. We find that all three models can generate monotonicity in the slope coefficients and R^2's over the forecast horizon. Furthermore, the model with priced parameter uncertainty produces dramatically larger (in absolute terms) slopes and R^2's relative to the AU approach and especially to the model with

Table 3
Return Predictability

h	Data Slope	Data R^2	PPU Slope	PPU R^2	AU Slope	AU R^2	FI Slope	FI R^2
Panel A: Investment-capital ratio $(i-k)$								
1Y	−0.394	0.071	−0.504	0.037	−0.108	0.043	−0.090	0.035
2Y	−0.603	0.123	−0.910	0.071	−0.198	0.079	−0.165	0.065
3Y	−1.293	0.245	−1.287	0.106	−0.280	0.114	−0.236	0.095
4Y	−1.831	0.333	−1.639	0.138	−0.362	0.150	−0.307	0.126
5Y	−2.453	0.372	−1.970	0.163	−0.439	0.182	−0.376	0.155
Panel B: Tobin's Q								
1Y	−0.335	0.078	−0.493	0.052	−0.434	0.042	−0.362	0.035
2Y	−1.573	0.133	−0.875	0.096	−0.792	0.079	−0.661	0.065
3Y	−1.952	0.165	−1.228	0.137	−1.123	0.114	−0.945	0.095
4Y	−2.443	0.192	−1.578	0.179	−1.452	0.150	−1.232	0.126
5Y	−2.815	0.231	−1.907	0.214	−1.758	0.181	−1.507	0.155
Panel C: Dividend-price ratio $(d^l - p^l)$								
1Y	0.083	0.041	0.024	0.030	0.013	0.018	0.011	0.016
2Y	0.122	0.055	0.040	0.051	0.023	0.033	0.018	0.029
3Y	0.175	0.074	0.055	0.070	0.030	0.045	0.024	0.040
4Y	0.212	0.093	0.069	0.089	0.037	0.055	0.029	0.050
5Y	0.227	0.105	0.080	0.103	0.042	0.063	0.034	0.058
Panel D: Consumption-wealth ratio $(c-w)$								
1Y	3.173	0.086	2.050	0.085	1.788	0.071	2.110	0.039
2Y	5.944	0.182	3.429	0.138	3.195	0.124	3.717	0.067
3Y	7.845	0.274	4.620	0.184	4.412	0.171	5.159	0.093
4Y	9.352	0.297	5.821	0.229	5.627	0.217	6.659	0.122
5Y	11.134	0.327	6.933	0.269	6.747	0.259	8.110	0.148

This table reports univariate regressions of cumulative excess log equity returns on several valuation and macroeconomic variables over various forecasting horizons (h years; 1 to 5). We use investment-capital ratio, Tobin's Q, dividend-price and consumption-wealth ratios as the right-hand side variable (x_t) in the linear projection:

$$r^{ex}_{t+1\to t+h} = \text{Intercept} + \beta(h) \times x_t + \varepsilon_{t+h},$$

where $r^{ex}_{t+1\to t+h}$ are h-year future excess log equity returns. The empirical statistics are for the U.S. data from 1952:Q1 to 2016:Q4. The PPU column refers to the production economy with rational pricing of parameter uncertainty, whereas the AU column refers to the production economy with AU pricing. In both cases, parameter uncertainty includes unknown transition probabilities and mean growth rates. The FI column presents the results of the full information case where the parameters are known. For each model, we simulate 1,000 economies at a quarterly frequency with a sample size equal to the empirical counterpart. We obtain the slope coefficients and R^2's for each simulation and report average sample statistics over all 1,000 artificial series.

known parameters.

3.4. Pricing a Claim to Calibrated Dividends

In the previous section, we studied the implications of parameter uncertainty for the dynamics of macroeconomic quantities and equity returns. We introduced financial leverage in the spirit of Jermann (1998) in order to make the equity return more risky. The main drawback of financial leverage considered in the previous section is that the leveraged firm's dividends still remained significantly procyclical in the model and, thus, the equity premium and its volatility were too small compared to the data. Furthermore, following the discussion of Kaltenbrunner and Lochstoer (2010), one can argue that the aggregate stock market dividends are only a small part of the payouts of the productive sector and, thus, cannot be directly interpreted as the firm's dividends in our model. Therefore, we follow a consumption-based asset pricing literature by directly calibrating an exogenous dividend process to replicate the stock market dividends.

Following Bansal and Yaron (2004), we price a levered consumption claim with a leverage factor λ. We formally define quarterly log dividend growth as follows:

$$\Delta d_t^M = g_d + \lambda \Delta c_t + \sigma_d \varepsilon_t^d, \tag{10}$$

where $\varepsilon_t^d \overset{\text{iid}}{\sim} N(0,1)$, g_d and σ_d are the dividend growth rate and volatility, respectively. We calibrate the parameters g_d, σ_d, and λ to make model implied statistics of dividend growth consistent with the historical data. Panel B in Table 1 reports the parameter values in an exogenous dividend stream. We set the mean adjustment (g_d) and the idiosyncratic dividend volatility (σ_d) to match the observed annual mean growth (2.06 percent) and volatility (10.38 percent) of dividends for the considered period. The leverage parameter (λ) is equal to 3.5, a midpoint of the range from 2.5 to 4.5 used in other studies.

Let R_{t+1}^M denote the return on a claim delivering stochastic dividends given by (10). Then:

$$R_{t+1}^M = \frac{P_{t+1}^M + D_{t+1}^M}{P_t^M} = \frac{P_{t+1}^M / D_{t+1}^M + 1}{P_t^M / D_t^M} \cdot \frac{D_{t+1}^M}{D_t^M}.$$

Substituting this expression into the equilibrium condition (5), the price-dividend ratio of a claim on the aggregate stock market dividends satisfies the equation:

$$\frac{P_t^M}{D_t^M} = E_t \left[M_{t+1} \left(1 + \frac{P_{t+1}^M}{D_{t+1}^M} \right) \frac{D_{t+1}^M}{D_t^M} \right]. \tag{11}$$

Unconditional Moments. Now we take a closer look at the equity claim paying stochastic dividends as a leverage on consumption similarly to Bansal and Yaron (2004).

The numerical methods used to solve for the equilibrium price-dividend ratio are presented in the Appendix. Table 4 shows the model-implied statistics of dividend growth, excess equity returns, the Sharpe ratio and the price-dividend ratio.

The calibrated dividends closely replicate the empirical first and second moments as well as a positive correlation between dividends and consumption observed in the data. Our conservative choice of a leverage parameter produces a slightly lower correlation between dividend and consumption growth rates, but it is crucial that the correlation remains positive in all models. Turning to equity moments, parameter uncertainty with AU pricing produces similar results to the production model with known parameters. Relative to the FI and AU cases, a priced parameter uncertainty approach significantly improves the fit of the model with the data. The model with parameter uncertainty and a prior sample of learning of 100 years match the sample equity premium, its volatility, the equity Sharpe ratio and the level of the price-dividend ratio well. Furthermore, the volatility of the price-dividend ratio comes out two to three times its value with fixed parameters, though it still remains lower than in the data. In the data, the log price-dividend ratio is highly persistent and the model with parameter learning reconciles this feature. Furthermore, looking at the results based on different training samples, one can see that Bayesian learning and rational pricing of an investor's subjective beliefs generates permanent shocks in the production economy.

It is important to stress that the implications of parameter learning in the production-based setting are based on a productivity growth process that is estimated over the post-war data. Even though the parameter estimates in our model reflect the business cycle fluctuations rather than rare and bad macroeconomic events, learning about the true productivity growth process has significant quantitative effects. This is mainly due to the fact that the impact of endogenous long-run risks originating from belief revisions is magnified by long-run risks in consumption growth through consumption smoothing, as documented by Kaltenbrunner and Lochstoer (2010). In the consumption-based setting, in order to match the financial moments one needs to add either learning about rare events observed in the pre-war data (Collin-Dufresne, Johannes and Lochstoer, 2016) or a more complex learning process (Johannes, Lochstoer and Mou, 2016). Furthermore, given *confounding* effects[5] documented by Johannes, Lochstoer and Mou (2016), we expect that

[5] *Confounding* effectively means that uncertainty about one variable makes learning about another variable more difficult.

Table 4
Calibrated Stock Market Dividend Claim

	Data	PPU				AU	FI
		100 yrs	150 yrs	200 yrs	∞ yrs		
$E(\Delta d^M)$	2.06	1.89	1.80	1.75	1.68	1.60	1.60
$\sigma(\Delta d^M)$	10.38	11.48	11.51	11.52	11.54	11.54	11.56
$ar1(\Delta d^M)$	0.25	0.01	0.01	0.01	0.01	0.01	0.01
$\rho(\Delta c, \Delta d^M)$	0.44	0.32	0.32	0.32	0.32	0.32	0.32
$E(R^M - R_f)$	5.51	5.92	5.18	4.64	3.75	2.88	2.80
$\sigma(R^M - R_f)$	16.55	15.90	15.43	15.18	14.78	14.81	14.42
$SR(R^M - R_f)$	0.33	0.33	0.27	0.24	0.20	0.15	0.15
$E(p^M - d^M)$	3.19	3.15	3.28	3.37	3.54	3.76	3.78
$\sigma(p^M - d^M)$	0.33	0.07	0.05	0.04	0.03	0.05	0.02
$ar1(p^M - d^M)$	0.97	0.90	0.87	0.84	0.79	0.90	0.76

This table reports the average moments from 1,000 simulations of 260 quarters of the data from the production economy considered in this paper, where the transition probabilities and mean growth rates are assumed to be unknown. As in Bansal and Yaron (2004), equity is a claim to an exogenous dividend stream. The historical data moments are reported in the data column and correspond to the U.S. data from 1952:Q1 to 2016:Q4. The PPU column refers to the production economy with rational pricing of parameter uncertainty, whereas the AU column refers to the production economy with AU pricing. In both cases, parameter uncertainty includes unknown transition probabilities and mean growth rates. The FI column presents the results of the full information case where the parameters are known. $E(x)$ and $\sigma(x)$ denote the average sample mean and standard deviations of x, respectively. $ar1(x)$ and $\rho(x, y)$ denote the average sample autocorrelation of x and correlation between x and y, respectively. All statistics are expressed in annualized terms, except for correlations and autocorrelations expressed in quarterly terms.

learning additionally about volatility risks and especially introducing a multidimensional learning problem with model, state and parameter uncertainty are expected to slow down the speed of learning and, thus, will improve our results. Finally, adding a more rare state into the productivity growth is expected to amplify the impact of parameter uncertainty, following the results of Collin-Dufresne, Johannes and Lochstoer (2016). We view the investigation of these Bayesian approaches as an interesting avenue for future research.

Conditional Dynamics. Figure 2 plots the responses of several key variables to a bad state realization, that lasts for 1 quarter, 3 quarters and 2 years. The sharp decline in beliefs about the probability of staying in the good state leads to a reduction in the interest rate, a decline in the price-dividend ratio as well as an increase in the risk premium and equity volatility. As long as the economy stays in the low productivity growth regime, the agent learns about the persistence of the bad state by revising his beliefs upward. During this period, the interest rates are low, the price-dividend ratio keeps declining, while

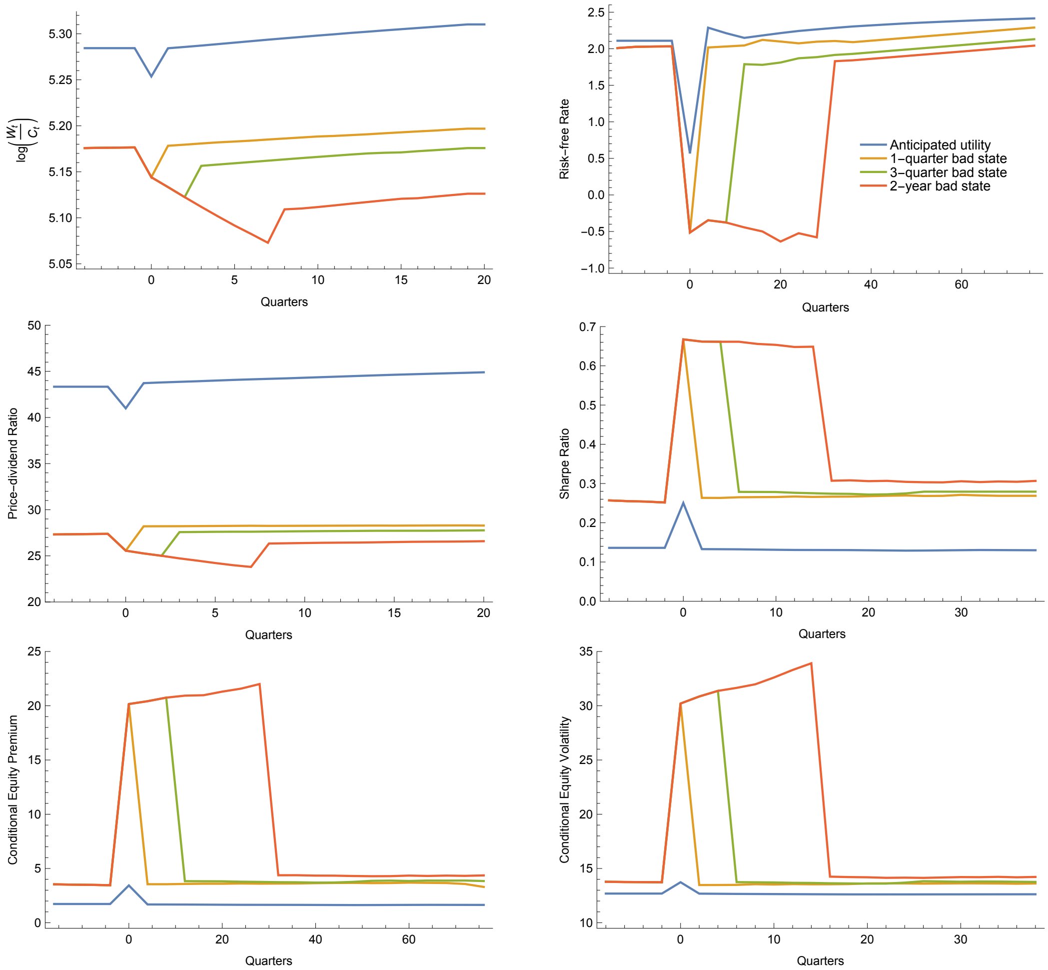

Figure 2: Conditional Prices and Moments. This figure shows the conditional risk-free rate, the price-dividend and equity Sharpe ratios, as well as the conditional equity premium and its volatility. The simulated variables are impulse response functions to the realization of a bad state of 1 quarter, 3 quarters and 2 years in the production economy, considered in this paper for the case of a 100-year prior. The economy is assumed to stay in the high productivity growth steady-state for a long period, and the representative agent holds unbiased initial mean beliefs. We report the conditional dynamics of the variables for the AU and PPU cases. For the sake of a convenient exposition, the former one includes only the responses to a 1-quarter bad state realization. The Appendix describes the numerical approach used.

the equity Sharpe ratio, the conditional equity premium and volatility remain elevated. Although both AU and PPU pricing predict similar paths of financial variables in response to a negative long-run risk shock to the expected productivity growth, the magnitude of their responses is substantially different.

For the anticipated utility case, one can observe very moderate responses in the returns, prices and conditional moments upon the onset of the bad state. Before the regime switch, both the conditional equity premium and the conditional Sharpe ratio are too low relative to the data and then they approximately double in response to the negative shock. Meanwhile, the conditional equity volatility increases only marginally in this case. In contrast, rationally priced parameter uncertainty predicts around 6-fold and 3-fold increases in the conditional risk premium and the equity Sharpe ratio, respectively. The equity volatility turns out to be highly countercyclical as it increases by a factor of about 2.5. The interest rate drops more in bad times with parameter uncertainty, while the realized equity returns are more volatile. The price-dividend ratio experiences about the same percentage decline upon the realization of the low productivity growth state in both cases. However, the level of the price-dividend ratio is substantially higher with AU, while parameter learning generates reasonable levels of the price-dividend ratio.

3.5. Adding Costly Reversibility

In the previous section we demonstrated that parameter learning and rational pricing are able to reproduce salient features of the macroeconomic quantities and equity returns, as long as the dividends exhibit a positive correlation with business cycle. However, it is important to maintain the endogeneity of the dividend process within the model. To fix this issue, we present an extension of the model where we include investment frictions in the form of costly reversibility. This will endogenously generate more procyclical dividends consistent with the data.

Formally, we model costly reversibility by adopting the asymmetric capital adjustment cost function, which takes a quadratic form:

$$\varphi(x_t) = x_t - \frac{\theta_t}{2} \cdot (x_t - x_0)^2 ,$$

where

$$\theta_t = \theta^+ \cdot \mathbb{I}(x_t \geq x_0) + \theta^- \cdot \mathbb{I}(x_t < x_0)$$

and $\mathbb{I}(\cdot)$ denotes the indicator operator that equals 1 if the condition is satisfied and 0 otherwise. We choose the constant x_0 such that there are no adjustment costs in the non-stochastic steady state, which implies $x_0 = \exp(\bar{\mu}) - 1 + \delta$. The remaining two parameters θ^+ and θ^- satisfy the condition $0 < \theta^+ < \theta^-$ to capture the idea of costly reversibility: the representative firm faces higher capital adjustment costs for the investment decisions leading to the capital stock being below a non-stochastic steady state value. In the quantitative exercise here, we calibrate the parameters θ^+ and θ^- consistent with the literature. The empirical estimates of θ^+ vary from 2 to 8. We choose a middle point of this range and set $\theta^+ = 5$ as in Zhang (2005). We further follow Zhang (2005) by assuming the degree of asymmetry equal $\theta^-/\theta^+ = 10$ that would imply $\theta^- = 50$.

Table 5 shows the results of the model with the asymmetric adjustment cost calibrated above and other parameters fixed at the values in Table 1. As shown in Panel A of Table 5, the model generates volatility of macroeconomic quantities relatively close to the data, though consumption is slightly more volatile and investment is smoother compared to the case with a convex adjustment cost function. Also, the quadratic adjustment cost better matches the comovements between macroeconomic variables than the convex adjustment cost.

Panel B summarizes the model-generated statistics of financial variables. Our calibration with investment reversibility predicts dividend dynamics quite similar to the data. Most importantly, the correlation between consumption and dividends becomes slightly positive, which in turn has a large impact on equity returns. In particular, the unconditional risk premium compares quite well with the sample estimate. Even though the excess volatility puzzle remains unresolved, the rationally priced parameter uncertainty magnifies the unconditional second moment of excess equity returns compared to the full information case and, in turn, explains around two thirds of the equity volatility in the data. Further, the mean, volatility and autocorrelation of the log price-dividend ratio compare surprisingly well to the observed point estimates. The introduction of additional channels such as, for example, a combination of wage rigidity and a constant elasticity of substitution (CES) production function (Favilukis and Lin, 2016) or learning about time-varying volatility risks can further improve the model performance; however, we leave a rigorous investigation of a more complex model for future research.

4. Sensitivity Analysis

We examine the sensitivity of our results to the choice of the two parameters (ψ, ξ), which determine how the agent is willing to substitute consumption intertemporally and how the capital stock can be adjusted over time. These channels are the two natural candidates to influence the propagation of and the interplay between the productivity shocks and subjective long-run risks due to Bayesian learning in our model. Table 6 presents a two-part sensitivity analysis by decreasing the EIS and considering different capital adjustment cost parameters, while keeping other values as in the benchmark calibration. For convenience, we report the results of the simulations for the setting with PPU and AU pricing based on a 100-year prior period.

Table 5
Sample Moments: The Extended Model with Costly Reversibility

	Data	PPU				AU	FI
		100 yrs	150 yrs	200 yrs	∞ yrs		
Panel A: Macroeconomic Quantities							
$\sigma(\Delta c)$	1.26	1.59	1.60	1.61	1.62	1.61	1.65
$\sigma(\Delta i)$	4.51	3.31	3.29	3.27	3.25	3.30	3.23
$\sigma(\Delta y)$	2.41	1.97	1.97	1.97	1.97	1.97	1.97
$ar1(\Delta c)$	0.32	0.18	0.18	0.18	0.18	0.18	0.18
$\rho(\Delta i, \Delta y)$	0.72	0.90	0.90	0.90	0.90	0.89	0.88
$\rho(\Delta c, \Delta y)$	0.52	0.85	0.85	0.85	0.85	0.85	0.85
$\rho(\Delta c, \Delta i)$	0.36	0.55	0.55	0.55	0.55	0.52	0.49
Panel B: Financial Variables							
$E(R_f) - 1$	1.44	1.64	1.77	1.85	1.98	2.02	2.03
$\sigma(R_f)$	1.07	0.59	0.55	0.53	0.49	0.44	0.44
$\sigma(M)/E(M)$		0.28	0.26	0.25	0.21	0.19	0.19
$E(\Delta d^l)$	2.06	0.77	0.88	0.95	1.05	1.14	1.13
$\sigma(\Delta d^l)$	10.38	9.98	10.77	11.34	12.55	15.12	14.40
$ar1(\Delta d^l)$	0.25	−0.06	−0.06	−0.06	−0.06	−0.06	−0.06
$\rho(\Delta c, \Delta d^l)$	0.44	0.01	0.01	0.02	0.03	0.03	0.07
$E(R^l - R_f)$	5.51	5.71	5.14	4.53	3.59	3.50	3.29
$\sigma(R^l - R_f)$	16.55	11.04	10.73	9.88	8.56	9.80	8.93
$E(p^l - d^l)$	3.19	2.89	2.97	3.06	3.23	3.29	3.32
$\sigma(p^l - d^l)$	0.33	0.26	0.26	0.26	0.26	0.30	0.29
$ar1(p^l - d^l)$	0.97	0.92	0.92	0.92	0.92	0.92	0.92

This table reports the average moments from 1,000 simulations of 260 quarters of the data from the production economy considered in this paper, where the transition probabilities and mean growth rates are assumed to be unknown. The historical data moments are reported in the data column and correspond to the U.S. data from 1952:Q1 to 2016:Q4. The PPU column refers to the production economy with rational pricing of parameter uncertainty, whereas the AU column refers to the production economy with AU pricing. In both cases, parameter uncertainty includes unknown transition probabilities and mean growth rates. The FI column presents the results of the full information case where the parameters are known. $E(x)$ and $\sigma(x)$ denote the average sample mean and standard deviations of x, respectively. $ar1(x)$ and $\rho(x, y)$ denote the average sample autocorrelation of x and correlation between x and y, respectively. All statistics are expressed in annualized terms, except for market price of risk given in percent, whereas correlations and autocorrelations are expressed in quarterly terms.

The elasticity of intertemporal substitution is an important parameter for matching the moments of macroeconomic variables as shown in Panel A of Table 6. For both PPU and AU, a lower EIS reduces the volatility of investment growth and makes consumption growth more volatile relative to the benchmark calibration. This is due to the fact that the investor is less willing to substitute consumption intertemporally. The AU columns of Panel A also suggest that the EIS does not affect the correlations between macroeconomic quantities. Interestingly, the PPU case predicts significantly different moments of macroeconomic variables for the smaller EIS. Indeed, consumption, investment and output become less correlated mainly due to larger short-run risks in consumption growth.

The bottom panel of Table 6 shows the impact of the EIS on financial moments. As expected, the risk-free rate is inversely related to the EIS parameter. The prices of equity on the endogenous levered firm's payouts are not markedly affected by the EIS. Turning to the equity claim on aggregate market dividends, there are large differences in the average equity premium and equity volatility. In this case, more volatile consumption growth predicts riskier dividends, which are modeled as a leverage on consumption. Therefore, the equity as a leveraged consumption claim implies the lower price-dividend ratios, the higher equity premium and equity volatility for a smaller value of the EIS. Notably, this impact on the financial moment is magnified in the PPU case.

As an additional exercise, we change the degree of capital adjustment costs in the production economy. The lower values of ξ introduce higher costs for capital adjustment. Decreasing the value of ξ to 2.5 leads to more volatile consumption growth and smoother investment growth. This additionally generates more volatile Tobin's Q and increases the mean and volatility of the investment return in the model. In general, the higher capital adjustment costs generate stronger short-run risks in the model and reduce the impact of the long-run risks generated by Bayesian learning and rational belief pricing. Since the latter shocks are the dominant drivers of high and volatile equity returns in the model, a claim to aggregate dividends becomes less risky as reflected in the higher price-dividend ratio, lower equity premium and equity volatility. In the light of this observation, a lower capital adjustment cost helps jointly match salient moments of macroeconomic quantities and financial returns. In particular, increasing the value of ξ to 5.5 moves the model-implied volatilities of consumption and investment closer to the data. Most importantly, parameter uncertainty generates stronger propagation of productivity shocks in this case by lowering the correlations between macroeconomic variables. In addition, stronger long-run risks originating from rational belief pricing further lead to a substantial increase in risk premia compared to the AU case, as evidenced in the last two columns of Table 6.

To assess the impact of asymmetric adjustment costs, we conduct sensitivity analysis to alternative parameter choices of θ^+ and θ^-. Table 7 shows that the model with symmetric quadratic adjustment costs $\theta^+ = \theta^- = 5$ displays a small equity premium and levered equity volatility originating from a wrong business cycle movement of dividends.

Table 6
Sensitivity Analysis

	$\psi = 1.2$		$\psi = 1.5$		$\xi = 2.5$		$\xi = 5.5$	
	PPU	AU	PPU	AU	PPU	AU	PPU	AU
Panel A: Macroeconomic Quantities								
$\sigma(\Delta c)$	1.54	1.51	1.44	1.43	1.46	1.49	1.21	1.22
$\sigma(\Delta i)$	2.99	3.00	3.20	3.18	3.13	3.12	3.74	3.68
$\sigma(\Delta y)$	1.95	1.94	1.96	1.94	1.96	1.96	1.95	1.93
$\rho(\Delta i, \Delta y)$	0.91	0.98	0.96	0.99	0.99	0.98	0.97	0.99
$\rho(\Delta c, \Delta y)$	0.93	0.98	0.96	0.99	0.99	0.98	0.92	0.98
$\rho(\Delta c, \Delta i)$	0.69	0.92	0.83	0.97	0.97	0.92	0.79	0.96
Panel B: Financial Variables								
$E(R_f) - 1$	1.96	2.52	1.84	2.34	1.64	2.10	1.64	2.17
$\sigma(R_f)$	0.43	0.37	0.42	0.35	0.48	0.37	0.36	0.28
$E(R^l - R_f)$	2.86	1.59	2.89	1.67	3.69	2.23	2.55	1.50
$\sigma(R^l - R_f)$	5.23	4.72	5.35	4.72	6.98	6.12	4.64	4.04
$E(R^M - R_f)$	9.90	3.55	8.03	3.31	5.60	3.06	6.56	2.82
$\sigma(R^M - R_f)$	21.41	15.98	18.50	15.41	15.39	14.98	16.57	14.75
$SR(R^M - R_f)$	0.40	0.19	0.36	0.17	0.30	0.16	0.35	0.15
$E(p^M - d^M)$	2.64	3.45	2.84	3.56	3.22	3.71	3.10	3.78
$\sigma(p^M - d^M)$	0.14	0.05	0.11	0.05	0.06	0.05	0.08	0.05
$ar1(p^M - d^M)$	0.90	0.84	0.90	0.86	0.90	0.90	0.90	0.90

This table reports the average moments from 1,000 simulations of 260 quarters of the data from the production economy considered in this paper, where the transition probabilities and mean growth rates are assumed to be unknown. The PPU column refers to the production economy with rational pricing of parameter uncertainty, whereas the AU column refers to the production economy with AU pricing. In both cases, parameter uncertainty includes unknown transition probabilities and mean growth rates. $E(x)$ and $\sigma(x)$ denote the average sample mean and standard deviations of x, respectively. $ar1(x)$ and $\rho(x, y)$ denote the average sample autocorrelation of x and correlation between x and y, respectively. All statistics are expressed in annualized terms, except for correlations and autocorrelations expressed in quarterly terms.

This result mimics our findings in the benchmark model with convex adjustment costs. Introducing costly reversibility improves the asset pricing implications of the model. We quantify this improvement by varying the degree of asymmetry. In particular, we fix $\theta^+ = 4$ and consider three cases for $\theta^-/\theta^+ : 10,\ 12.5$ and 15. Table 7 reports that a higer degree of asymmetry increases the average mean and volatility of levered equity returns. This comes as a result of a more procyclical firm's dividends consistent with our main result: the importance of parameter learning in the production economy is conditional on the introduction of procyclical dividends in the economy.

Table 7
Sensitivity Analysis: The Extended Model with Costly Reversibility

	$\theta^+ = 5$ $\theta^- = 5$		$\theta^+ = 4$ $\theta^- = 40$		$\theta^+ = 4$ $\theta^- = 50$		$\theta^+ = 4$ $\theta^- = 60$	
	PPU	AU	PPU	AU	PPU	AU	PPU	AU
Panel A: Macroeconomic Quantities								
$\sigma(\Delta c)$	1.14	1.12	1.51	1.54	1.58	1.60	1.63	1.64
$\sigma(\Delta i)$	3.98	3.91	3.42	3.41	3.36	3.34	3.27	3.22
$\sigma(\Delta y)$	1.93	1.92	1.97	1.97	1.97	1.96	1.97	1.96
$\rho(\Delta i, \Delta y)$	0.91	0.99	0.93	0.92	0.91	0.89	0.90	0.89
$\rho(\Delta c, \Delta y)$	0.78	0.96	0.86	0.84	0.83	0.83	0.84	0.84
$\rho(\Delta c, \Delta i)$	0.45	0.91	0.62	0.57	0.53	0.50	0.53	0.52
Panel B: Financial Variables								
$E(R_f) - 1$	1.62	2.23	1.66	2.03	1.65	2.00	1.62	2.00
$\sigma(R_f)$	0.28	0.24	0.55	0.43	0.59	0.45	0.59	0.45
$E(\Delta d^l)$	0.58	1.24	0.72	1.13	0.76	1.14	0.77	1.12
$\sigma(\Delta d^l)$	24.45	27.45	11.29	16.72	10.39	15.65	9.86	15.11
$ar1(\Delta d^l)$	−0.06	−0.03	−0.04	−0.05	−0.06	−0.06	−0.06	−0.05
$\rho(\Delta c, \Delta d^l)$	−0.11	−0.61	−0.10	−0.05	0.01	0.05	0.06	0.08
$E(R^l - R_f)$	2.03	1.15	4.88	3.00	5.68	3.53	6.49	3.82
$\sigma(R^l - R_f)$	3.65	3.21	9.32	8.09	11.02	9.93	13.25	10.90

This table reports the average moments from 1,000 simulations of 260 quarters of the data from the production economy considered in this paper, where the transition probabilities and mean growth rates are assumed to be unknown. The PPU column refers to the production economy with rational pricing of parameter uncertainty, whereas the AU column refers to the production economy with AU pricing. In both cases, parameter uncertainty includes unknown transition probabilities and mean growth rates. $E(x)$ and $\sigma(x)$ denote the average sample mean and standard deviations of x, respectively. $ar1(x)$ and $\rho(x, y)$ denote the average sample autocorrelation of x and correlation between x and y, respectively. All statistics are expressed in annualized terms, except for correlations and autocorrelations expressed in quarterly terms.

5. Conclusion

In this paper, we show that introducing rational parameter learning into an otherwise standard real business cycle model improves its ability to match asset return data. The model with priced parameter uncertainty has a small effect on the second moments of macroeconomic variables and more significant impact on the comovements between quantities. Parameter learning generates a substantial amplification of the risk premium on a levered firm's payouts and reproduces the long-horizon predictability of excess returns by macroeconomic and valuation variables. Furthermore, we show that rational belief pricing considered in this paper has the largest impact on equity returns when introducing and pricing a procyclical dividend growth process. In this case, the production economy can closely replicate the first and second moments of risk-free rates and excess equity returns, the equity Sharpe ratio and the level of the price-dividend ratio, while generating smooth consumption and volatile investment. Finally, we show that introducing investment friction in the form of costly reversibility helps endogenously generate a pro-cyclical dividend process and, at the same time, to maintain all the desired pricing effects that come from multidimensional learning and rational pricing.

Future research may consider extending our mechanism to a richer model with sticky prices and financial frictions. In particular, modeling wage rigidity in the spirit of Favilukis and Lin (2016) can help endogenously generate procyclical dividend growth in the model. The interaction between sticky prices and learning effects may have additional interesting implications for the labor market. Motivated by a large strand of the literature on time-varying macroeconomic uncertainty, it is interesting and straightforward to extend our methodology to learning about volatility risks. This might have additional asset pricing implications, especially for volatility sensitive assets, as well as interesting effects for the real economy.

Appendix

A. Numerical Algorithm: Anticipated Utility

In the AU case, the representative household learns about the unknown parameters by updating his beliefs upon the realization of new data, but ignores parameter uncertainty when making decisions. Thus, although the beliefs vary over time, the household centers the "true" parameters at the current posterior means and keeps these subjective estimates constant while solving for the continuation utility (and a levered equity claim) in each period.

In this paper, we focus on two learning about parameters economies with unknown transition probabilities, and unknown transition probabilities and mean growth rates.[6] The numerical solution for both models under AU pricing simplifies to solving for the equilibrium pricing ratios when all parameters are actually known by the household. We find the solution of these simplest economies on a dense grid for unknown parameters (that is, unknown transition probabilities in the former model; unknown transition probabilities and mean growth rates in the latter model). Then the household uses these equilibrium pricing functions for the decision making and asset pricing based on the current beliefs.

A.1. All Known Parameters

Productivity growth is given by:

$$\Delta a_t = \mu_{s_t} + \sigma\varepsilon_t,$$

where $\varepsilon_t \overset{\text{iid}}{\sim} N(0,1)$, s_t is a two state Markov chain with transition matrix:

$$\Pi = \begin{bmatrix} \pi_{11} & 1-\pi_{11} \\ 1-\pi_{22} & \pi_{22} \end{bmatrix},$$

where $\pi_{ii} \in (0,1)$. The regimes switches in s_t are independent of the Gaussian shocks ε_t.

Here, we give details on how the continuation utility (and a levered equity claim) is computed for the economy with all parameters known. We define the following stationary variables:

$$\left\{\tilde{C}_t, \tilde{I}_t, \tilde{Y}_t, \tilde{K}_t, \tilde{U}_t\right\} = \left\{\frac{C_t}{A_t}, \frac{I_t}{A_t}, \frac{Y_t}{A_t}, \frac{K_t}{A_t}, \frac{U_t}{A_t}\right\}$$

[6] The methodology for the AU case (as well as the priced parameter uncertainty case in Appendix B) can be further extended for learning about the volatility of productivity growth. However, we leave this investigation for the future research.

The household's problem is:

$$\tilde{U}_t = \max_{\tilde{C}_t, \tilde{I}_t} \left\{ (1-\beta)\tilde{C}_t^{1-\frac{1}{\psi}} + \beta \left(E_t \left[\tilde{U}_{t+1}^{1-\gamma} \cdot \left(\frac{A_{t+1}}{A_t} \right)^{1-\gamma} \right] \right)^{\frac{1-\frac{1}{\psi}}{1-\gamma}} \right\}^{\frac{1}{1-\psi}} \tag{A1}$$

subject to the constraints:

$$\tilde{C}_t + \tilde{I}_t = \tilde{K}_t^{\alpha} \bar{N}^{1-\alpha} \tag{A2}$$

$$e^{\Delta a_{t+1}} \tilde{K}_{t+1} = (1-\delta)\tilde{K}_t + \varphi \left(\frac{\tilde{I}_t}{\tilde{K}_t} \right) \tilde{K}_t \tag{A3}$$

$$\Delta a_t = \mu_{s_t} + \sigma \varepsilon_t, \;\; \varepsilon_t \sim N(0,1) \tag{A4}$$

$$\tilde{C}_t \geq 0, \;\; \tilde{K}_{t+1} \geq 0 \tag{A5}$$

where the subscript t indicates the time, $E_t(\cdot)$ denotes the expectation conditional on the information available at time t. Because the parameters are assumed known, s_t and $\tilde{K}_t$ are the only state variables in the economy. Ultimately, the recursive equation (A1) can be rewritten as:

$$\begin{aligned} & \tilde{U}_t(s_t, \tilde{K}_t) \\ = & \max_{\tilde{C}_t, \tilde{I}_t} \left\{ (1-\beta)\tilde{C}_t^{1-\frac{1}{\psi}} + \beta \left(E_t \left[\tilde{U}_{t+1} \left(s_{t+1}, \tilde{K}_{t+1} \right)^{1-\gamma} \cdot e^{(1-\gamma)\Delta a_{t+1}} \right] \right)^{\frac{1-\frac{1}{\psi}}{1-\gamma}} \right\}^{\frac{1}{1-\psi}} \end{aligned} \tag{A6}$$

To solve the recursion (A6), we use the the value function iteration algorithm. In particular, the numerical algorithm proceeds as follows:

1. We find the de-trended steady state capital $\tilde{K}_{ss}$, assuming the productivity growth equals the steady state level predicted by a Markov-switching model. The state space for capital normalized by technology is set at $[0.2\tilde{K}_{ss}, 2.2\tilde{K}_{ss}]$. We further use $n_k = 100$ points on a grid for capital in the numerical computation. A denser grid does not lead to significantly different results.

2. For any level of capital $\tilde{K}_t$ at time t, we construct a grid for $\tilde{I}_t$ with uniformly distributed points between 0 and $\tilde{K}_t^{\alpha} \bar{N}^{1-\alpha}$. Specifically, we use $n_i = 400$ points.

3. For the expectation, we use the Gauss-Hermite quadrature with $n_{gh} = 8$ points. Using the quadrature weights and nodes, we can calculate the expression on the right hand side.

4. We solve the optimization problem in the Bellman equation (A6) subject to (A2)-(A5) and update a new value function $\tilde{U}_t = \tilde{U}_t(s_t, \tilde{K}_t)$ given an old one $\tilde{U}_{t+1} = \tilde{U}_{t+1}(s_{t+1}, \tilde{K}_{t+1})$.

5. We iterate Steps 2-4 by updating the continuation utility on each iteration until a suitable convergence is achieved. Specifically, the stopping rule is that the distance between the new value function and the old value function satisfies $|\tilde{U}_{t+1} - \tilde{U}_t|/|\tilde{U}_t| < 10^{-12}$.

B. Numerical Algorithm: Priced Parameter Uncertainty

The numerical solution for the case of priced parameter uncertainty consists of two main steps[7]. First, we solve for the equilibrium pricing ratios when true parameters are actually known by the household (by assumption, these are learned at $T = \infty$). We find the solution of this simplest limiting economy on a dense grid of state variables. Second, we use the known parameters boundary economies as terminal values in the backward recursion to obtain the equilibrium function at time t. For the first step, Appendix A outlines details of the numerical algorithm for all known parameters. Therefore, we present the solution methodology employed at the second step for two models with unknown transition probabilities, and unknown transition probabilities and mean growth rates.

B.1. Unknown Transition Probabilities

Productivity growth is given by:

$$\Delta a_t = \mu_{s_t} + \sigma\varepsilon_t,$$

where $\varepsilon_t \overset{\text{iid}}{\sim} N(0,1)$, s_t is a two state Markov chain with a transition matrix:

$$\Pi = \begin{bmatrix} \pi_{11} & 1-\pi_{11} \\ 1-\pi_{22} & \pi_{22} \end{bmatrix},$$

where $\pi_{ii} \in (0,1)$. The regimes switches in s_t are independent of the Gaussian shocks ε_t.

In the case of unknown transition probabilities, the representative household knows true values of the parameters within each state (μ_1, μ_2, σ) and observes states (s_t) but

[7] Johnson (2007) uses this solution methodology in a case with parameter learning and power utility. Johannes, Lochstoer and Mou (2016) and Collin-Dufresne, Johannes and Lochstoer (2016) extend this approach to the case of Epstein-Zin utility in the endowment economy. We further extend the numerical solution to the case of Epstein-Zin utility in the production economy.

does not know the transition probabilities (π_{11}, π_{22}). At time $t = 0$, the household holds priors about uncertain probabilities in the transition matrix and updates beliefs each period upon realization of new series and regimes. We assume a Beta distributed prior and, thus, posterior beliefs are also Beta distributed.

The Beta distribution has the probability density function of the form:

$$p(\pi|a,b) = \frac{\pi^{a-1}(1-\pi)^{b-1}}{B(a,b)},$$

where $B(a,b)$ is the Beta function (a normalization constant), a and b are two positive shape parameters. We are particularly interested in the expected value of the Beta distribution defined by:

$$E[\pi|a,b] = \frac{a}{a+b}.$$

Furthermore, we use two pairs of hyperparameters parameters (a_1, b_1) and (a_2, b_2) for unknown transition probabilities in the states π_{11} and π_{22}, respectively. At time t, the household uses Bayes' rule and the fact that states are observable to update hyperparameters for each state i as follows:

$$a_{i,t} = a_{i,0} + \#(\text{state } i \text{ has been followed by state } i), \tag{B7}$$

$$b_{i,t} = b_{i,0} + \#(\text{state } i \text{ has been followed by state } j), \tag{B8}$$

given the initial prior beliefs $a_{i,0}$ and $b_{i,0}$.

Once we find the limiting boundary economies on the first step, we perform a backward recursion using the following state variables:

$$\tau_{1,t} = a_{1,t} - a_{1,0} + b_{1,t} - b_{1,0} \tag{B9}$$

$$\lambda_{1,t} = E_t[\pi_{11}] = \frac{a_{1,t}}{a_{1,t} + b_{1,t}} \tag{B10}$$

$$\tau_{2,t} = a_{2,t} - a_{2,0} + b_{2,t} - b_{2,0} \tag{B11}$$

$$\lambda_{2,t} = E_t[\pi_{22}] = \frac{a_{2,t}}{a_{2,t} + b_{2,t}} \tag{B12}$$

Note that $X_t = \{\tau_{1,t}, \lambda_{1,t}, \tau_{2,t}, \lambda_{2,t}\}$ are sufficient statistics for the agent's priors. Also, we can update X_{t+1} using the equations (B7)-(B12), the next period regime, and sufficient statistics:

$$X_{t+1} = f(s_{t+1}, s_t, X_t).$$

For notational purposes, it might be useful to denote $X_t^s \equiv \{\tau_{1,t}, \lambda_{1,t}, \tau_{2,t}, \lambda_{2,t}\}$ and $X_t^{\Delta a} \equiv \{\tilde{K}_t\}$, where the superscripts s and Δa indicate that variables in the vectors X_t^s and $X_t^{\Delta a}$

are a function only of the observed state realization s_t and a function of (also) the realized productivity growth, respectively. Thus, $X_t = \left[X_t^s, X_t^{\Delta a}\right]$. Using these notations, we can rewrite

$$\tilde{U}_{t+1}(s_{t+1}, X_{t+1}) = \tilde{U}_{t+1}(s_{t+1}, s_t, X_t^s, \Delta a_{t+1}, X_t^{\Delta a})$$

to better indicate the dependence of state variables on specific shocks. Ultimately, the recursive equation (A1) can be rewritten as:

$$\begin{aligned} &\tilde{U}_t(s_t, X_t) \qquad \text{(B13)} \\ = \; &\max_{\tilde{C}_t, \tilde{I}_t} \left\{ (1-\beta)\tilde{C}_t^{1-\frac{1}{\psi}} + \beta \left(E_t \left[\tilde{U}_{t+1}^{1-\gamma}\left(s_{t+1}, s_t, X_t^s, \Delta a_{t+1}, X_t^{\Delta a}\right) \cdot e^{(1-\gamma)\Delta a_{t+1}} \middle| s_t, X_t \right] \right)^{\frac{1-\frac{1}{\psi}}{1-\gamma}} \right\}^{\frac{1}{1-\psi}}, \end{aligned}$$

where the expectation on the right hand side is equivalent to:

$$\begin{aligned} & E_t \left[\tilde{U}_{t+1}^{1-\gamma}\left(s_{t+1}, s_t, X_t^s, \Delta a_{t+1}, X_t^{\Delta a}\right) \cdot e^{(1-\gamma)\Delta a_{t+1}} \middle| s_t, X_t \right] \\ = \; & E_t \left[E_t \left[\tilde{U}_{t+1}^{1-\gamma}\left(s_{t+1}, s_t, X_t^s, \Delta a_{t+1}, X_t^{\Delta a}\right) \cdot e^{(1-\gamma)\Delta a_{t+1}} \middle| s_{t+1}, s_t, X_t \right] \middle| s_t, X_t \right] \\ = \; & \sum_{s_{t+1}=1}^{2} \mathbb{P}(s_{t+1}|s_t, X_t^s) \quad \ldots \\ \times \; & E_t \left[\tilde{U}_{t+1}^{1-\gamma}\left(s_{t+1}, s_t, X_t^s, \Delta a_{t+1}, X_t^{\Delta a}\right) \cdot e^{(1-\gamma)\Delta a_{t+1}} \middle| s_{t+1}, s_t, X_t \right] \\ = \; & \sum_{s_{t+1}=1}^{2} E_t(\pi_{s_{t+1},s_t}|s_t, X_t^s) \quad \ldots \\ \times \; & E_t \left[\tilde{U}_{t+1}^{1-\gamma}\left(s_{t+1}, s_t, X_t^s, \Delta a_{t+1}, X_t^{\Delta a}\right) \cdot e^{(1-\gamma)\Delta a_{t+1}} \middle| s_{t+1}, s_t, X_t \right], \qquad \text{(B14)} \end{aligned}$$

where the first and second equalities follow from the independency of the regime changes and the Gaussian shocks to productivity growth (s_{t+1} and ε_{t+1}). Let the conditional density of π_{s_{t+1},s_t} be $g(\pi_{s_{t+1},s_t}|s_t, X_t^s)$, then the third equality follows from:

$$\mathbb{P}(s_{t+1}|s_t, X_t^s) = \int_0^1 \pi_{s_{t+1},s_t} g(\pi_{s_{t+1},s_t}|s_t, X_t) d\pi_{s_{t+1},s_t} = E_t(\pi_{s_{t+1},s_t}|s_t, X_t)$$

Furthermore, using the definition of our state variables, this last conditional expectation equals $\lambda_{s_t,t}$ or $1 - \lambda_{s_t,t}$.

Note that before choosing the optimal consumption and investment in (B13), we need to solve numerically first the inner expectation, which is equivalently represented by (B14). Hopefully, we have an analytical expression for the conditional expectation of transition probabilities in (B14), which is either $\lambda_{s_t,t}$ or $1-\lambda_{s_t,t}$. For the second conditional expectation in (B14), we do not have a closed form since the continuation utility depends on the realized productivity growth through $\tilde{K}_{t+1}$. Therefore, we use quadrature-type numerical methods to evaluate this expectation as follows:

$$\begin{aligned} & E_t\left[\tilde{U}_{t+1}^{1-\gamma}\left(s_{t+1}, s_t, X_t^s, \Delta a_{t+1}, X_t^{\Delta a}\right) \cdot e^{(1-\gamma)\Delta a_{t+1}} \Big| s_{t+1}, s_t, X_t\right] \\ \approx & \sum_{j=1}^{J} \omega_\varepsilon(j)\left[\tilde{U}_{t+1}^{1-\gamma}\left(s_{t+1}, s_t, X_t^s, \Delta a(j), X_t^{\Delta a}\right) \cdot e^{(1-\gamma)\Delta a(j)} \Big| s_{t+1}, s_t, X_t\right], \end{aligned} \tag{B15}$$

where $\omega_\varepsilon(j)$ is the quadrature weight corresponding to the quadrature node $n_\varepsilon(j)$ used for the integration of a standard normal shock ε_{t+1} in productivity growth. The observed realized productivity growth, $\Delta a(j)$, and a state variable, $X_{t+1}^{\Delta a}(j) = \tilde{K}_{t+1}(j)$, are updated as follows:

$$\Delta a(j) = \mu_{s_{t+1}} + \sigma \cdot n_\varepsilon(j) \tag{B16}$$

$$e^{\Delta a(j)}\tilde{K}_{t+1}(j) = (1-\delta)\tilde{K}_t + \varphi\left(\frac{\tilde{I}_t}{\tilde{K}_t}\right)\tilde{K}_t, \tag{B17}$$

where

$$\tilde{I}_t = \tilde{K}_t^\alpha \bar{N}^{1-\alpha} - \tilde{C}_t. \tag{B18}$$

Finally, the numerical backward recursion can be performed by using (B13)-(B18). The boundary conditions are defined by the limiting economies $\tau_{1,\infty}$ and $\tau_{2,\infty}$, where the transition probabilities π_{11} and π_{22} are known.

B.1.1. Solving for a Dividend Claim

We also solve for the price-dividend ratio of the equity claim written on aggregate dividends, which are defined as a leverage to aggregate consumption. Let exogenous aggregate dividends be given by:

$$\Delta d_{t+1} = g_d + \lambda \Delta c_{t+1} + \sigma_d \varepsilon_{d,t+1},$$

where $g_d = \Big(1-\lambda\Big)\Big(E(\mathbb{P}(s_\infty = 1|\pi_{11}, \pi_{22}))\mu_1 + E(\mathbb{P}(s_\infty = 2|\pi_{11}, \pi_{22}))\mu_2\Big)$ and $\mathbb{P}(s_\infty = i|\pi_{11}, \pi_{22})$ is the ergodic probability of being in state i conditional on the transition probabilities π_{11} and π_{22}. Note that the long run mean of dividends growth, g_d, is changing under the household's filtration, though the true long run growth is constant. The subjective beliefs about the true parameter values induce fluctuations in g_d, which can be expressed as $g_d = g_d(s_{t+1}, s_t, X_t)$.

The equilibrium condition for the price-dividend ratio is standard in the Epstein-Zin economy and is given by:

$$PD_t = E_t\left[\beta\left(\frac{\tilde{C}_{t+1}}{\tilde{C}_t}\right)^{-\frac{1}{\psi}}\left(\frac{A_{t+1}}{A_t}\right)^{-\frac{1}{\psi}}\left(\frac{\tilde{U}_{t+1}\cdot\left(\frac{A_{t+1}}{A_t}\right)}{\mathcal{R}_t\left(\tilde{U}_{t+1}\cdot\left(\frac{A_{t+1}}{A_t}\right)\right)}\right)^{\frac{1}{\psi}-\gamma}\left(\frac{D_{t+1}}{D_t}\right)(PD_{t+1}+1)\right] \tag{B19}$$

Similarly to the solution for the value function, we rewrite all variables in the recursion (B19) as a function of the state variables and further use quadrature-type numerical methods to evaluate expectations on the right hand side of (B19). Additionally, we update the long run dividends growth, $g_d(s_{t+1}, s_t, X_t)$, which is in fact random. Consequently, the equilibrium recursion used to solve the model is then:

$$\begin{aligned}
& PD_t(s_t, X_t,) \\
= \; & E_t\left[\begin{array}{c} \beta e^{\left(\lambda-\frac{1}{\psi}\right)(\Delta\tilde{c}_{t+1}+\Delta a_{t+1})}\left(\frac{\tilde{U}_{t+1}\cdot e^{\Delta a_{t+1}}}{\mathcal{R}_t\left(\tilde{U}_{t+1}\cdot e^{\Delta a_{t+1}}\right)}\right)^{\frac{1}{\psi}-\gamma}\cdots \\ \times \quad e^{g_d(s_{t+1},s_t,X_t)+0.5\sigma_d^2}\cdot\left(PD_{t+1}\left(s_{t+1}, s_t, X_t^s, \Delta a_{t+1}, X_t^{\Delta a}\right)+1\right)\end{array}\middle| s_t, X_t\right] \\
= \; & E_t\left[E_t\left[\begin{array}{c} \beta e^{\left(\lambda-\frac{1}{\psi}\right)(\Delta\tilde{c}_{t+1}+\Delta a_{t+1})}\left(\frac{\tilde{U}_{t+1}\cdot e^{\Delta a_{t+1}}}{\mathcal{R}_t\left(\tilde{U}_{t+1}\cdot e^{\Delta a_{t+1}}\right)}\right)^{\frac{1}{\psi}-\gamma}\cdots \\ \times \quad e^{g_d(s_{t+1},s_t,X_t)+0.5\sigma_d^2}\cdot\left(PD_{t+1}\left(s_{t+1}, s_t, X_t^s, \Delta a_{t+1}, X_t^{\Delta a}\right)+1\right)\end{array}\middle| s_{t+1}, s_t, X_t\right]\middle| s_t, X_t\right] \\
= \; & \sum_{s_{t+1}=1}^{2}\mathbb{P}(s_{t+1}|s_t, X_t^s) \quad \cdots \\
\times \; & E_t\left[\begin{array}{c} \beta e^{\left(\lambda-\frac{1}{\psi}\right)(\Delta\tilde{c}_{t+1}+\Delta a_{t+1})}\left(\frac{\tilde{U}_{t+1}\cdot e^{\Delta a_{t+1}}}{\mathcal{R}_t\left(\tilde{U}_{t+1}\cdot e^{\Delta a_{t+1}}\right)}\right)^{\frac{1}{\psi}-\gamma}\cdots \\ \times \quad e^{g_d(s_{t+1},s_t,X_t)+0.5\sigma_d^2}\cdot\left(PD_{t+1}\left(s_{t+1}, s_t, X_t^s, \Delta a_{t+1}, X_t^{\Delta a}\right)+1\right)\end{array}\middle| s_{t+1}, s_t, X_t\right] \\
= \; & \sum_{s_{t+1}=1}^{2}E_t(\pi_{s_{t+1},s_t}|s_t, X_t^s) \quad \cdots \\
\times \; & E_t\left[\begin{array}{c} \beta e^{\left(\lambda-\frac{1}{\psi}\right)(\Delta\tilde{c}_{t+1}+\Delta a_{t+1})}\left(\frac{\tilde{U}_{t+1}\cdot e^{\Delta a_{t+1}}}{\mathcal{R}_t\left(\tilde{U}_{t+1}\cdot e^{\Delta a_{t+1}}\right)}\right)^{\frac{1}{\psi}-\gamma}\cdots \\ \times \quad e^{g_d(s_{t+1},s_t,X_t)+0.5\sigma_d^2}\cdot\left(PD_{t+1}\left(s_{t+1}, s_t, X_t^s, \Delta a_{t+1}, X_t^{\Delta a}\right)+1\right)\end{array}\middle| s_{t+1}, s_t, X_t\right]
\end{aligned}$$

Again, the conditional expectation of transition probabilities under the household's filtration permits an analytical formula, while the inner expectation in the expression above can be evaluated using the quadrature-type integration methods.

B.1.2. Limiting Economies - Boundary Values for General Case

The key assumption of the numerical solution is that the household eventually learns the true values of all uncertain parameters in the productivity growth. Thus, the simplest limiting economy is the one where all parameters are known, including both transition probabilities π_{11} and π_{22}. In this case, s_t and K_t are the only state variables in the economy. We employ the numerical solution methodology outlined for AU pricing for this limiting economy. Specifically, we find the continuation utility (and the price-dividend ratio of the equity claim) for a grid on π_{11} and π_{22}.

B.2. Unknown Transition Probabilities and Unknown Mean Growth Rates

Productivity growth is given by:

$$\Delta a_t = \mu_{s_t} + \sigma \varepsilon_t,$$

where $\varepsilon_t \overset{\text{iid}}{\sim} N(0, 1)$, s_t is a two state Markov chain with the transition matrix:

$$\Pi = \begin{bmatrix} \pi_{11} & 1 - \pi_{11} \\ 1 - \pi_{22} & \pi_{22} \end{bmatrix},$$

where $\pi_{ii} \in (0, 1)$. The regimes switches in s_t are independent of the Gaussian shocks ε_t.

As before, we assume that the representative household does not know the transition probabilities (π_{11}, π_{22}). Additionally, the mean growth rates within each state (μ_1, μ_2) are assumed to be unknown, while the realization of states (s_t) and productivity volatility (σ_t) remain observable. Due to the limitations of the numerical solution algorithm under the prices parameter uncertainty case, we are unable to extend the economy to unobservable regimes, while it is still possible to assume that the household does not know a volatility parameter. Nevertheless, the extension to the case with all parameters unknown, including volatility except for states, is quite straightforward, and we leave the investigation of learning about volatility parameters for future research.

Regarding priors, we assume a conjugate prior for transition probabilities and mean growth rates within each state i : the Beta distributed prior and the truncated normal distributed prior, respectively. The updating equations for two pairs of hyperparameters (a_1, b_1) and (a_2, b_2) remain as before. Additionally, we denote hyperparameters of the truncated normal distributed prior for mean growth in state i by $\mu_{i,t}$ and $\sigma_{i,t}$, which are updated by the Bayes' rule as follows:

$$\mu_{i,t+1} = \mu_{i,t} + \mathbf{1}_{s_{t+1}=i} \frac{\sigma_{i,t}^2}{\sigma_i^2 + \sigma_{i,t}^2} \left(\Delta a_{t+1} - \mu_{i,t}\right) \tag{B20}$$

$$\sigma_{i,t+1}^{-2} = \mathbf{1}_{s_{t+1}=i} \cdot \sigma_i^{-2} + \sigma_{i,t}^{-2}, \tag{B21}$$

where $\mathbf{1}$ is an indicator function that equals 1 if the condition in subscript is true and 0 otherwise.

Note that since the variance hyperparameters $\sigma_{1,t}^2$ and $\sigma_{2,t}^2$ are a function of the time, the following 6-dimensional vector $X_t \equiv \{\tau_{1,t}, \lambda_{1,t}, \tau_{2,t}, \lambda_{2,t}, \mu_{1,t}, \mu_{2,t}\}$ is sufficient statistics for the priors. Thus, we can define X_{t+1} using the equations (B7)-(B12), (B20)-(B21), the next period regime, and sufficient statistics at time t :

$$X_{t+1} = f(s_{t+1}, s_t, X_t).$$

Following the notations of a previous section, we define $X_t^s \equiv \{\tau_{1,t}, \lambda_{1,t}, \tau_{2,t}, \lambda_{2,t}\}$ and $X_t^{\Delta a} \equiv \{\tilde{K}_t, \mu_{1,t}, \mu_{2,t}\}$, where the superscripts s and Δa indicate that variables in the vectors X_t^s and $X_t^{\Delta a}$ are a function only of the observed state realization s_t and a function of (also) the realized productivity growth, respectively. Thus, $X_t = \left[X_t^s, X_t^{\Delta a}\right]$. Using these notations, we can rewrite

$$\tilde{U}_{t+1}(s_{t+1}, X_{t+1}) = \tilde{U}_{t+1}(s_{t+1}, s_t, X_t^s, \Delta a_{t+1}, X_t^{\Delta a})$$

to better indicate the dependence of state variables on specific shocks. Ultimately, the recursive equation (A1) is of the same form:

$$\begin{aligned} & \tilde{U}_t(s_t, X_t) \qquad \text{(B22)} \\ = \; & \max_{\tilde{C}_t, \tilde{I}_t} \left\{ (1-\beta)\tilde{C}_t^{1-\frac{1}{\psi}} + \beta \left(E_t \left[\tilde{U}_{t+1}^{1-\gamma} \left(s_{t+1}, s_t, X_t^s, \Delta a_{t+1}, X_t^{\Delta a}\right) \cdot e^{(1-\gamma)\Delta a_{t+1}} \middle| s_t, X_t \right] \right)^{\frac{1-\frac{1}{\psi}}{1-\gamma}} \right\}^{\frac{1}{1-\psi}}, \end{aligned}$$

where the expectation on the right hand side is equivalent to:

$$\begin{aligned} & E_t \left[\tilde{U}_{t+1}^{1-\gamma} \left(s_{t+1}, s_t, X_t^s, \Delta a_{t+1}, X_t^{\Delta a}\right) \cdot e^{(1-\gamma)\Delta a_{t+1}} \middle| s_t, X_t \right] \\ = & \sum_{s_{t+1}=1}^{2} E_t(\pi_{s_{t+1},s_t} | s_t, X_t^s) \quad \ldots \\ \times & E_t \left[\tilde{U}_{t+1}^{1-\gamma} \left(s_{t+1}, s_t, X_t^s, \Delta a_{t+1}, X_t^{\Delta a}\right) \cdot e^{(1-\gamma)\Delta a_{t+1}} \middle| s_{t+1}, s_t, X_t \right]. \qquad \text{(B23)} \end{aligned}$$

In this case, we compute the conditional expectation in (B23) by integrating over conditional distribution of mean growth rates as well as Gaussian distribution of the error term in productivity growth. In particular:

$$
\begin{aligned}
& E_t\left[\tilde{U}_{t+1}^{1-\gamma}\left(s_{t+1}, s_t, X_t^s, \Delta a_{t+1}, X_t^{\Delta a}\right) \cdot e^{(1-\gamma)\Delta a_{t+1}} \middle| s_{t+1}, s_t, X_t\right] \\
\approx \quad & \sum_{j=1}^{J} \omega_\varepsilon(j) \left[\sum_{k=1}^{K} \omega_{\mu_{s_{t+1}}}(k) \cdot \tilde{U}_{t+1}^{1-\gamma}\left(s_{t+1}, s_t, X_t^s, \Delta a(j,k), X_t^{\Delta a}\right) \cdot e^{(1-\gamma)\Delta a(j,k)} \middle| s_{t+1}, s_t, X_t\right],
\end{aligned}
\tag{B24}
$$

where $\omega_\varepsilon(j)$ is the quadrature weight corresponding to the quadrature node $n_\varepsilon(j)$ used for the integration of a standard normal shock ε_{t+1} in productivity growth, and $\omega_{\mu_{s_{t+1}}}(k)$ is the quadrature weight corresponding to the quadrature node $n_{\mu_{s_{t+1}}}(k)$ used for the integration of a truncated standard normal variable $\mu_{s_{t+1}}$. The observed realized productivity growth, $\Delta a(j,k)$, and a state variable, $X_{t+1}^{\Delta a}(j,k) = \tilde{K}_{t+1}(j,k)$, are updated as follows:

$$
\Delta a(j,k) = n_{\mu_{s_{t+1}}}(k) + \sigma \cdot n_\varepsilon(j) \tag{B25}
$$

$$
e^{\Delta a(j,k)} \tilde{K}_{t+1}(j,k) = (1-\delta)\tilde{K}_t + \varphi\left(\frac{\tilde{I}_t}{\tilde{K}_t}\right)\tilde{K}_t, \tag{B26}
$$

where

$$
\tilde{I}_t = \tilde{K}_t^\alpha \bar{N}^{1-\alpha} - \tilde{C}_t. \tag{B27}
$$

Finally, the numerical backward recursion can be performed by using (B22)-(B27). The boundary conditions are defined by the limiting economies $\tau_{1,\infty}$ and $\tau_{2,\infty}$, where the transition probabilities π_{11} and π_{22}, and mean growth rates μ_1 and μ_2, are known.

B.2.1. Solving for a Dividend Claim

We also solve for the price-dividend ratio of the equity claim written on aggregate dividends, which are defined as a leverage to aggregate consumption. Let exogenous aggregate dividends be given by:

$$
\Delta d_{t+1} = g_d + \lambda \Delta c_{t+1} + \sigma_d \varepsilon_{d,t+1},
$$

where $g_d = \Big(1-\lambda\Big)\Big(E(\mathbb{P}(s_\infty = 1|\pi_{11}, \pi_{22}))\mu_1 + E(\mathbb{P}(s_\infty = 2|\pi_{11}, \pi_{22}))\mu_2\Big)$ and $\mathbb{P}(s_\infty = i|\pi_{11}, \pi_{22})$ is the ergodic probability of being in state i conditional on the transition probabilities π_{11} and π_{22}.

Note that the long run mean of dividends growth, g_d, is changing under the household's filtration, though the true long run growth is constant. The subjective beliefs about the true parameter values induce fluctuations in g_d, which can be expressed as $g_d = g_d(s_{t+1}, s_t, X_t)$. The equilibrium condition for the price-dividend ratio and the equilibrium recursion remain the same as in the "unknown transition probabilities" model. The only difference between the two models lie in the way we calculate the conditional expectations. With unknown transition probabilities and mean growth rates in the productivity growth process, we employ quadrature-type integration methods analogous to solving for the continuation utility in this economy.

B.2.2. Limiting Economies - Boundary Values for General Case

The key assumption of the numerical solution is that the household eventually learns the true values of all uncertain parameters in the productivity growth. Thus, the simplest limiting economy is the one where all parameters are known, including both transition probabilities π_{11} and π_{22}, mean growth rates μ_1 and μ_2. In this case, s_t and K_t are the only state variables in the economy. We employ the numerical solution methodology outlined for AU pricing for this limiting economy. Specifically, we find the continuation utility (and the price-dividend ratio of the equity claim) for a grid on $\pi_{11}, \pi_{22}, \mu_1$ and μ_2.

B.3. Existence of Equilibrium

Similarly to Collin-Dufresne, Johannes and Lochstoer (2016) and Johannes, Lochstoer and Mou (2016), the existence of the equilibrium in our production-based economy relies on the fact that the value function is concave and finite for all parameters known economies. Therefore, we verify that these conditions are satisfied for all limiting boundary economies.

C. Impulse Responses

In this section, we consider the numerical procedure used to obtain impulse responses of key macroeconomic and financial variables to a regime switch in the mean growth rate of productivity. In particular, we assume that the economy stays in the high growth state for a long period and then moves to a low growth regime at time 0. We further consider three possible scenarios where the economy remains in the bad regime for one quarter, three quarters, or two years before returning to the good state. The details of the numerical algorithm look as follows.

First, we find the steady state of capital, $\tilde{K}$, in the high growth regime, $s_t = 1$, assuming unbiased parameter beliefs, X_t, which are centered at the true values. Formally, $\tilde{K}$ solves the equation:

$$\tilde{K} = f^k(s_{-1} = 1, X_{-1}, \tilde{K}),$$

where $f^K(\cdot)$ is the policy function for capital assuming the productivity growth is high forever.

Second, suppose that the economy starts in the high growth steady state before time 0 and the investor holds unbiased parameter beliefs. Then unexpectedly the economy shifts to to the bad state at time 0 and stays there for τ periods. Using the policy function, capital is computed recursively as:

$$\begin{aligned}
\tilde{K}_{-1} &= \tilde{K}, \\
\tilde{K}_0 &= f^k(s_0 = 2, X_0, \tilde{K}_{-1}), ... \\
\tilde{K}_\tau &= f^k(s_\tau = 2, X_\tau, \tilde{K}_{\tau-1}), \\
\tilde{K}_{\tau+1} &= f^k(s_{\tau+1} = 1, X_{\tau+1}, \tilde{K}_\tau), ... \\
\tilde{K}_t &= f^k(s_t = 1, X_t, \tilde{K}_{t-1}), \quad \forall t,
\end{aligned}$$

where investor's parameter beliefs are updated in each period.

Third, we use policy functions for investment and consumption to obtain equilibrium values of $\tilde{I}_t$ and $\tilde{C}_t$. Finally, we calculate the remaining macroeconomic and financial variables using the updated state variables.

References

Abel, A, B. and Eberly, J. C. (1994), 'A unified model of investment under uncertainty', *American Economic Review* **84**, 1369–1384.

Abel, A, B. and Eberly, J. C. (1996), 'Optimal investment with costly reversibility', *Review of Economic Studies* **63**, 581–593.

Ai, H., Croce, M. and Li, K. (2013), 'Toward a quantitative general equilibrium asset pricing model with intangible capital', *Review of Financial Studies* **26**, 491–530.

Bansal, R., Kiku, D., Shaliastovich, I. and Yaron, A. (2014), 'Volatility, the macroeconomy and asset prices', *Journal of Finance* **69**, 2471–2511.

Bansal, R. and Yaron, A. (2004), 'Risks for the long run: A potential resolution of asset pricing puzzles', *The Journal of Finance* **59 (4)**, 1481–1509.

Barro, R. J. (2006), 'Rare disasters and asset markets in the twentieth century', *The Quarterly Journal of Economics* **121 (3)**, 823–866.

Basu, S. and Bundick, B. (2017), 'Uncertainty shocks in a model of effective demand', *Econometrica* **85 (3)**, 937–958.

Belo, F., Lin, X. and Bazdresch, S. (2014), 'Labor hiring, investment and stock return predictability in the cross section', *Journal of Political Economy* **122**, 129–77.

Bloom, N. (2009), 'The impact of uncertainty shocks', *Econometrica* **77**, 623–685.

Bloom, N., Floetotto, M., Jaimovich, N., Saporta-Eksten, I. and Terry, S. (2018), 'Really uncertain business cycles', *Forthcoming Econometrica* .

Boldrin, M., Christiano, L. J. and Fisher, J. D. M. (2001), 'Habit persistence, asset returns, and the business cycle', *American Economic Review* **91**, 149–66.

Born, B. and Pfeifer, J. (2014), 'Policy risk and the business cycle', *Journal of Monetary Economics* **68**, 68–85.

Cagetti, M., Hansen, L. P., Sargent, T. and Williams, N. (2002), 'Robustness and pricing with uncertain growth', *The Review of Financial Studies* **15**(2), 363–404.

Campanale, C., Castro, R. and Clementi, G. (2010), 'Asset pricing in a production economy with chewdekel preferences', *Review of Economic Dynamics* **13**, 379–402.

Campbell, J. and Shiller, R. (1988), 'The dividend-price ratio and expectations of future dividends and discount factors', *Review of Financial Studies* **1**, 195–228.

Christiano, L., Motto, R. and Rostagno, M. (2014), 'Risk shocks', *The American Economic Review* **104**(1), 27–65.

Cochrane, J. (1991), 'Production-based asset pricing and the link between stock returns and economic fluctuations', *Journal of Finance* **46**, 209–237.

Collin-Dufresne, P., Johannes, M. and Lochstoer, L. A. (2016), 'Parameter learning in general equilibrium: The asset pricing implications', *American Economic Review* **106 (3)**, 664–98.

Croce, M. M. (2014), 'Long-run productivity risk: A new hope for production-based asset pricing?', *Journal of Monetary Economics* **66**, 13–31.

Epstein, L. G. and Zin, S. E. (1989), 'Substitution, risk aversion, and the temporal behavior of consumption and asset returns: A theoretical framework', *Econometrica: Journal of the Econometric Society* **57 (4)**, 937–969.

Fama, E. F. and French, K. R. (1989), 'Business conditions and expected returns on stocks and bonds', *Journal of Financial Economics* **25**, 23–49.

Favilukis, J. and Lin, X. (2016), 'Wage rigidity: A quantitative solution to several asset pricing puzzles', *Review of Financial Studies* **29 (1)**, 148–192.

Fernandez-Villaverde, J., Guerron-Quintana, P., Rubio-Ramirez, J. and Uribe, M. (2011), 'Risk matters: the real effects of volatility shocks', *American Economic Review* **101**, 2530–2561.

Gilchrist, S., Sim, J. and Zakrajsek, E. (2014), 'Uncertainty, financial frictions, and irreversible investment', *Working Paper* .

Gourio, F. (2012), 'Disaster risk and business cycles', *American Economic Review* **102 (6)**, 2734–66.

Hall, R. E. (2001), 'The stock market and capital accumulation', *American Economic Review* **91**, 1185–1202.

Hamilton, J. (1989), 'A new approach to the economic analysis of nonstationary time series and the business cycle', *Econometrica* **57(2)**, 357–384.

Hamilton, J. (1990), 'Analysis of time series subject to changes in regimes', *Journal of Econometrics* **45**, 39–70.

Jahan-Parvar, M. and Liu, H. (2014), 'Ambiguity aversion and asset prices in production economies', *Review of Financial Studies* **27**, 3060–3097.

Jermann, U. J. (1998), 'Asset pricing in production economics', *Journal of Monetary Economics* **41**, 257–75.

Johannes, M., Lochstoer, L. A. and Mou, Y. (2016), 'Learning about consumption dynamics', *Journal of Finance* **71**, 551–600.

Johnson, T. (2007), 'Optimal learning and new technology bubbles', *Journal of Monetary Economics* **87**, 2486–2511.

Ju, N. and Miao, J. (2012), 'Ambiguity, learning, and asset returns', *Econometrica* **80**, 559–591.

Justiniano, A. and Primiceri, G. (2008), 'The time varying volatility of macroeconomic fluctuations', *American Economic Review* **98**, 604–641.

Kaltenbrunner, G. and Lochstoer, L. (2010), 'Long-run risk through consumption smoothing', *Review of Financial Studies* **23**, 3190–224.

Kydland, F. E. and Prescott, E. C. (1982), 'Time-to-build and aggregate fluctuations', *Econometrica* **50**, 1345–1370.

Leduc, S. and Liu, S. (2016), 'Uncertainty shocks are aggregate demand shocks', *Journal of Monetary Economics* **82**, 20–35.

Lettau, M. and Ludvigson, S. (2001), 'Consumption, aggregate wealth and stock returns', *Journal of Finance* **56**, 815–849.

Lewellen, J. (2004), ‘Predicting returns with financial ratios’, *Journal of Financial Economics* **74**, 209–35.

Liu, H. and Miao, J. (2014), ‘Growth uncertainty, generalized disappointment aversion and production-based asset pricing’, *Journal of Monetary Economics* **69**, 70–89.

Liu, H. and Miao, J. (2015), ‘Growth uncertainty, generalized disappointment aversion and production-based asset pricing’, *Journal of Monetary Economics* **69**, 70–89.

Long, J. B. and Plosser, C. I. (1983), ‘Real business cycles’, *Journal of Political Economy* **91**, 39–69.

Mehra, R. and Prescott, E. C. (1985), ‘The equity premium: a puzzle’, *Journal of Monetary Economics* **15**, 145–161.

Pontiff, J. and Schall, L. D. (1998), ‘Book-to-market ratios as predictors of market returns’, *Journal of Financial Economics* **49**, 141–60.

Restoy, F. and Rockinger, M. (1994), ‘On stock market returns and returns on investment’, *Journal of Finance* **49**, 543–56.

Rietz, T. A. (1988), ‘The equity risk premium: A solution’, *Journal of Monetary Economics* **22(1)**, 117–131.

Stock, J. and Watson, M. (1999), ‘Business cycle fluctuations in us macroeconomic time series’, *J. Taylor and M. Woodford (eds.), Handbook of Macroeconomics, Amsterdam, Elsevier* .

Tallarini, T. (2000), ‘Risk-sensitive real business cycles’, *Journal of Monetary Economics* **45**, 507–532.

Uhlig, H. (2007), ‘Explaining asset prices with external habits and wage rigidity in a dsge model’, *American Economic Review* **97**, 239–43.

Zhang, L. (2005), ‘The value premium’, *Journal of Finance* **60 (1)**, 67–103.

Abstrakt

Zkoumáme, jakým způsobem učení parametrů posiluje dopad makroekonomických šoků na ceny akcií a jiné veličiny ve standardní produkční ekonomice, kde má reprezentativní agent Epstein-Zin preference. Investor pozoruje technologické šoky, jejichž dynamika je dána procesem spromĕnlivým režimem, ale nezná skryté parametry modelu, které řídí krátkodobé a dlouhodobé vyhlídky ekonomického růstu. Ukazujeme, že racionáln učení parametrů endogenné generuje dlouhodobé riziko vekonomickém růstu a ve spotřebě, což pomáhá vysvětlit širokou škálu fenoménů dynamického oceňování aktiv. Implikace dlouhodobých subjektivních rizik pro oceňování aktiv zásadně závisí na zavedení pro-cyklického procesu dividend, který je konzistentní sdaty.

Working Paper Series
ISSN 1211-3298
Registration No. (Ministry of Culture): E 19443

Individual researchers, as well as the on-line and printed versions of the CERGE-EI Working Papers (including their dissemination) were supported from institutional support RVO 67985998 from Economics Institute of the CAS, v. v. i.

Specific research support and/or other grants the researchers/publications benefited from are acknowledged at the beginning of the Paper.

Published by
Charles University, Center for Economic Research and Graduate Education (CERGE)
and
Economics Institute of the CAS, v. v. i. (EI)
CERGE-EI, Politických vězňů 7, 111 21 Prague 1, tel.: +420 224 005 153, Czech Republic.
Printed by CERGE-EI, Prague
Subscription: CERGE-EI homepage: http://www.cerge-ei.cz

Phone: + 420 224 005 153
Email: office@cerge-ei.cz
Web: http://www.cerge-ei.cz

Editor: Byeongju Jeong

The paper is available online at http://www.cerge-ei.cz/publications/working_papers/.

ISBN 978-80-7343-447-2 (Univerzita Karlova, Centrum pro ekonomický výzkum a doktorské studium)
ISBN 978-80-7344-500-3 (Národohospodářský ústav AV ČR, v. v. i.)