THE OPTIMAL BALLOT STRUCTURE FOR DOUBLE-MEMBER DISTRICTS

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The Optimal Ballot Structure for Double-Member Districts*

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Abstract

The Anglo-American double-member districts employing plurality-at-large are frequently criticized for giving a large majority premium to a winning party. In this paper, we demonstrate that the premium stems from a limited degree of voters' discrimination associated with only two positive votes on the ballot. To enhance voters' ability to discriminate, we consider rules that give voters more positive and negative votes. We identify voting equilibria of alternative scoring rules in a situation where candidates differ in binary ideology and binary quality; strategic voters are of two ideology types; and a candidate's ideology is more salient than quality. The most generous rules such as approval voting and combined approval-disapproval voting only replicate the outcomes of plurality-at-large. The highest minority representation and the highest quality is achieved by a rule that assigns two positive votes and one negative vote to each voter.

Abstrakt

Volební výsledky dosažené ve volbách s dvoumandátovými obvody bývají často kritizovány jako velmi neproporční, protože vítězná strana získává vysokou prémii za své vítězství. V tomto textu ukazuji, že vysoká prémie je důsledkem omezené možnosti voličů diskriminovat mezi jednotlivými kandidáty, což není samo o sobě vlastností dvoumandátového obvodu, nýbrž hlasování na základě dvou kladných hlasů. Aby voliči mohli vice diskriminovat, musejí disponovat dodatečnými hlasy. Cílem tohoto textu je ukázat výsledky strategického hlasování pro volební pravidla ve dvoumandátových obvodech, která se liší počtem kladných a záporných hlasů, a to za situace, kdy kandidáti se liší v binární ideologii a binární kvalitě. Pravidla, která neomezují počty hlasů, dosahují pouze standardních výsledků. Nejlepší zastoupení menšinových voličů a nejvyšší kvalitu kandidátů dosahuje pravidlo, které dává voličům dva kladné hlasy a jeden záporný hlas.

Keywords: electoral rules, strategic voting, negative votes, plurality-at-large

JEL Classification: D72

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1 Introduction

In the year 2012, a Czech financier, mathematician and philanthropist Karel Janeček proposed a '2+1 electoral rule'.[1] This scoring electoral rule for double-member districts gives each voter two positive votes and a single negative vote, where the positive and negative points add up to each other. A voter cannot cumulate positive points to a single candidate, but partial abstention is allowed. A unique feature of the 2+1 electoral rule is the simultaneous presence of the positive and negative votes; hence, the rule blends structurally different properties of best-rewarding and worst-punishing electoral rules (see Myerson, 1999). Motivation for the rule is to allow voters to express their policy preferences safely but at the same time motivate them for discrimination along the quality (i.e., competence or integrity) dimension.

This paper attempts to compare the properties of the 2+1 rule relative to the properties of closely similar rules in two-member districts, including plurality-at-large that is currently almost exclusively used in Anglo-American double-member districts. Evidence from U.S. states demonstrates that plurality-at-large in two-member districts is associated with a low incidence of split outcomes (Cox, 1984). In fact, in the 2010–2012 elections in U.S. states, a single party gained both seats in 78.5% of two-member districts. The large premium for a majority party in a district is seen as the major disadvantage of having two-member districts as such, and district magnitude has even become a constitutional issue in the United States. In a famous 1986 decision, Thornburg v. Gingles, the U.S. Supreme Court overturned North Carolina's multi-member legislative lines on the grounds they discriminated against blacks. The U.S. Voting Rights Act thus encourages the creation of districts where racial or ethnic minorities predominate, and single-member districts are interpreted as best fitting this objective.

Our analysis of plurality-at-large reveals that disproportionality in two-member districts is the result of an excessively large correlation of scores of candidates coming from a single party. The correlation can be weakened by *reforming the structure of the ballot* towards endowing voters with more than just two positive votes. We demonstrate that the resulting outcomes improve not only representativeness, but also the quality of the elected representatives. For that purpose, we investigate several scoring rules for two-member districts including plurality-at-large, the 2+1 rule, approval voting, and the most generous rule known as combined approval-disapproval voting (Felsenthal, 1989). The rules differ in the maximal number of positive votes ($V^+ = 0,1,2,3$) and the maximal number of negative votes ($V^- = 0,1,2,3$).

We confine the analysis to an elementary electoral situation with exactly one valence dimension and one policy dimension, which contains the essential tradeoff between policy (private

[1] Source: `http://www.kareljanecek.com/muj-navrh-volebniho-zakona` (in Czech), accessed 16 November, 2012.

value) and quality (common value) as faced by instrumentally rational voters. We consider strategic voters[2] under population uncertainty. The set of strategic voters is driven from a probability distribution, and the optimal ballot is determined by the structure of decisive (pivotal) events. Pivotal events, where an individual vote can make a difference are essential for characterizing the optimal ballots.

We employ two models of population uncertainty. Drawing from multinomial tradition (Palfrey, 1989; Cox, 1994; Bouton, Castanheira and Llorente-Sauger, 2012; Carmona, 2012), we begin with a fully symmetric binomial setup. In that setup, all profiles for all electoral rules can be tractably analyzed without the need to calculate magnitudes of the pivotal events. Then, as a robustness check, we study the electoral rules in large asymmetric Poisson games, borrowing techniques from Myerson (2000, 2002), Bouton and Castanheira (2012), and Bouton (2013).

Consider two ex ante symmetric groups of voters (left-wing L-voters and right-wing R-voters). Total size is predetermined in the binomial game, and Poisson-distributed in the Poisson game. Each individual type is drawn independently and identically. As in Myerson (1993), each ideological type is represented by a single low-quality and a single high-quality candidate. Thus, the set of candidates K contains four generic types of candidates, differing in publicly known binary ideology and binary quality.

To motivate the analysis, we begin with the analysis of single-member districts.[3] The prevalent voting rule for single-member districts is *simple plurality*. By Duverger Law, a set of seriously competing candidates is restricted to a pair of candidates, hence to a single binary dimension, which may be a pure ideology dimension. Hence, the valence dimension may be entirely suppressed in the binary competition. Adding a single negative vote (*mixed 1+1 rule*) does not robustly solve the coordination problem since at pivotal events, the negative votes of one group tend to cancel out fully the positive votes of the other group. Consequently, the voters have to spread their few votes among very many serious candidates. The reason why the rule contains a well-known 'underdog' effect of pure negative voting (Myerson, 1999) is that it embeds too many negative votes relative to positive votes. Finally, adding an extra positive vote to the simple plurality rule, as in *approval voting* (AV), helps since casting a positive vote to the high-quality candidate from one's own group now possesses zero strategic risk (Myerson, 1993).

[2]The phenomenon of strategic or tactical voting has been identified in many contexts, including proportional and mixed systems, and ambiguity remains only over the size of the phenomenon. As an example, Kawai and Watanabe (2013) recently estimated a fully structural model of voting decisions in Japan's general election and concluded that between 63% and 85% of voters are strategic.

[3]The comparison rests on the idea that quality competition is present both in single-member and two-member districts; hence, four candidates run for office in both districts. If each group is represented by a single party, and each party nominates a single candidate, then valence competition is entirely absent in the elections, and the electoral outcome in a single-member district is invariant to the rule.

The intuition over the beneficial effect of extra positive votes and the detrimental effect of extra negative votes does not translate into two-member districts. We begin with a symmetric binomial setting. Under *plurality-at-large*, voters support their two ideological candidates independently on their qualities. As a result, the dominant voting group always elects their two candidates independently on quality. Approval voting does not change the equilibrium. Endowing the voter with an arbitrary number of both positive and negative votes as in *combined approval-disapproval voting* (CAV) does not help either because strategic voters tend to avoid ballots with zero points (Felsenthal, 1989; Núñez and Laslier, 2013).

Adding only a single negative vote (the 2+1 rule) now generates the following incentives: Each group of voters tries to win the two seats for their two candidates. In the electoral competition, two types of pivotal events arise: (i) If the group is in ex post majority (majority event), the stronger candidate of the group gains the first seat, and the weaker candidate of the group competes with the stronger candidate of the opposing group for the second seat. (ii) If the group is in ex post minority (minority event), the group cannot win the first seat, and its stronger candidate competes with the weaker candidate of the other group for the second seat. Since positive votes cannot accumulate to a single candidate, voters cast two positive votes for two candidates from their group to win both pivotal events. Quality then matters only for how the single negative vote is allocated.

The negative vote is cast to a low-quality candidate if the minority event turns out to be more valuable under a sincere profile. In the binomial setting, this case occurs for a large set of parameters. Intuitively, unless the probabilities of the majority and minority events differ too much from each other, only nominal win gains in the events matter. In the minority event, a nominal win gain is large because a win improves *both policy and quality*, whereas in the majority event, a nominal win is small because a win improves policy only at the expense of the lower quality of the elected candidate.

We compare the electoral outcomes achieved by plurality-at-large, approval voting, and combined approval-disapproval with the outcomes under the 2+1 rule in terms of quality and minority representation. The 2+1 rule increases the average quality of the elected candidates. By frequently generating split outcomes, it also protects minority interests more than the alternative rules in two-member districts. We additionally adopt a utilitarian welfare criterion to demonstrate that if the common valuation of quality is sufficiently large, utilitarian welfare unambiguously increases with the 2+1 rule.

To analyze the voters' incentives robustly, we adopt asymmetric large Poisson games. For the 2+1 rule, even under asymmetry, the ex-ante minority voters use the negative vote sincerely. The ex-ante majority voters strategically mix their negative votes to keep both opposing candidates equally serious. For the other electoral rules, pure ideology competition is the equilibrium for any parameters. Thus, we have again mixed electoral outcomes on the one side and non-mixed electoral outcomes on the other. Their comparison remains qualitatively similar to the binomial setting.

This paper adds to the very narrow literature on the simultaneous use of positive and negative votes that has to date been limited to the analysis of CAV. Second, similarly to runoff-voting analysis in Bouton (2013), the attention of strategic voting theory with population uncertainty is here devoted to the analysis of rules in multi-prize elections, not in single-member elections (Myerson, 2002; Myatt, 2007; Núñez, 2010; Krishna and Morgan, 2011; Bouton and Castanheira, 2012). The main difference is decisive races in multi-member districts are contests for the last seat not for the first seat; thus, pivotal events have a more complicated structure than in single-member districts. Most importantly, the seriousness of a candidate is not monotonic in her expected score. Third, the paper is novel in the comprehensive analysis of strategic voting equilibria for a set of four candidates and multiple electoral rules, and in the simultaneous analysis of the two most widely used population uncertainty models.[4] Most of the recent literature examines a single rule and only three candidates (e.g., Bouton and Castanheira, 2012), and adding any extra candidate increases exponentially the size of the strategy set.

The general lesson of the paper is that adding negative votes on the ballot is valuable for reflecting preferences down the ranking, but the voters must be motivated to submit a ballot with a rich structure that uses negative votes as a second-order degree of discrimination. To generate two degrees of discrimination by the equilibrium ballot, the number of positive votes V^+ and the number of negative votes V^- must be limited. Only this limit forces the voter to use the intermediate values (in our case, zero points). At the same time, too many levels of discrimination such as in Borda Count should be discouraged simply to avoid the strategic complexity in voting. The 2+1 rule is instrumental in capping the total number of votes ($V^+ + V^- < \#K$) and in limiting the structure of the votes ($V^- < V^+$).

The paper proceeds as follows. Section 2 motivates the electoral situation, electoral rules, and electoral behavior. Section 3 builds the symmetric binomial game. It analyzes three electoral rules for single-seat districts (plurality, mixed voting, and approval voting) and four electoral rules for two-seat districts (plurality-at-large, the 2+1 rule, approval voting, and combined approval-disapproval voting). Section 4 solves the equilibria for two-seat districts in a large asymmetric Poisson game. Section 5 concludes.

[4]An alternative that builds pivotal events in the absence of population uncertainty is noise in recording votes (score uncertainty), where each strategy profile is associated with a distribution of various outcomes (Laslier, 2009).

2 Electoral rules in two-member districts

Two-member districts were the norm in English elections from the thirteenth through most of the nineteenth century until massive redistricting in 1885. The American political system inherited its electoral laws from England, and the predominance of double- and other multi-member districts continued in the United States past the colonial period. The perspective of the time was more in favor of multi-member districts, where one of chief concerns with single-member districts was the excessive amount of special local legislation as observed by the New York Constitutional Commission in 1872 (Cox, 1984). Since World War II, apportionments nonetheless led to a gradual adoption of single-member districts across the United States.[5]

In political science, double-member districts used to be seen as a prospective remedy to the disproportionality in legislative representation associated with simple plurality. For Lijphart and Grofman (1984, p. 8), "*. . . a two-member district PR system could achieve the functional purpose of plurality even better than the plurality method itself.*" Recently, Carey and Hix (2011) revised the tradeoff associated with the district magnitude, and for a sample of elections from 1945 to 2006 in all democratic countries with a population of more than one million, they find an optimal district magnitude to be in the range of three to eight.

Currently, there are 9 assemblies in the U.S. states that use two-member districts and elect both representatives per district through plurality-at-large (dual voting, block voting). To demonstrate the performance of plurality-at-large, we calculate the shares of mixed (split) and non-mixed (non-split) electoral outcomes exploiting data from the most recent elections (2010–2012).[6]

In Table 1, the share of mixed outcomes is only 21.4–21.5% of all outcomes, measured either as the average from all districts or as the average of district averages. The low incidence of mixed outcomes suggests that replacing single-member districts by two-member districts may not improve representativeness. In theory, increasing the district magnitude improves representativeness of dispersed minorities at the expense of the representativeness of concentrated minorities. In the context of the U.S. states, the negative effect upon concentrated minorities seems

[5] Another largely discussed example of the two-seat system with two positive votes is the 'binomial' open-list-PR system in Chile. In Chile, the system was designed in the last years of the Pinochet regime with the aim to strategically overrepresent a coalition of right-wing parties relative to a center-left coalition. In the binomial system, if two parties are competing, the minority party gains a seat in the district (i.e., half of the total seats) unless it obtains less than a third of the votes. To achieve that level of representation was strategic also with respect to a two-third qualified majority required for constitutional change. Lacy and Niou (1998) provide an analysis relevant to strategic position-taking in the Chilean binomial system.

[6] Source: `http://ballotpedia.org/wiki/index.php/State_legislative_chambers_that_use_multi-member_districts`, accessed 2 February, 2013.

to outweigh the positive effect for dispersed minorities. The fact that two-member districts magnify disproportionality relative to single-member districts is seen as a major shortcoming of the two-member districts. Combined with the popular idea that single-member districts improve accountability, it is not surprising that reapportionments in many U.S. states are towards single-member districts.

Table 1: The shares of mixed electoral outcomes in two-member districts in recent U.S. states' elections

U.S. state	Election year	Assembly	Mixed	Total	Share
Arizona	2012	House of Representatives	2	28	7.1 %
Maryland	2010	House of Representatives	3	15	20 %
New Hampshire	2012	House of Representatives	17	53	32.1 %
New Jersey	2011	General Assembly	0	40	0 %
North Dakota	2012	House of Representatives	6	25	24 %
South Dakota	2012	House of Representatives	9	33	27.3 %
Vermont	2012	Senate	2	6	33.3 %
Vermont	2012	House of Representatives	14	45	31.1 %
West Virginia	2012	House of Delegates	2	11	18.2 %

Our paper aims to show that the source of disproportionality is primarily found in the low degree of discrimination available with the plurality-at-large rule not in the existence of the two-member districts. From our perspective, voters need multiple degrees of discrimination to incorporate secondary (here quality) considerations successfully into their ballots. In the class of scoring rules, this requires the simultaneous existence of both positive and negative votes.

In this paper, we conduct an elementary analysis of the electoral competition in two-member districts where we let candidates for office compete both in a conflicting (policy) dimension and in a non-conflicting (quality or valence) dimension.[7] The candidate's quality can also be interpreted as corruptability; hence, valence competition represents one of many channels between electoral rules and corruption (c.f., Persson, Tabellini and Trebbi, 2003).[8]

[7]In its broad definition, valence is used for any valuable personal characteristic including campaigning or networking skills, but in a narrow sense, valence is only for the qualities that voters value for their own sake such as integrity, competence, and dedication to public service. More specifically, Stone and Simas (2010) measure character-valence through seven indicators: personal integrity, an ability to work well with other leaders, an ability to find solutions to problems, competence, a grasp of the issues, qualifications to hold public office, and overall strength as a public servant.

[8]Many effects of the electoral system upon the level of corruption are related to incentives and disincentives to raise rents depending on the size of the party system and coalitional behavior. These effects can be modeled

We assume that both policy preferences and valence characteristics are predetermined and known. The reason is to isolate the pure and direct effect of the electoral rule on the electoral expectations for the high- and low-quality candidates and thereby on the calculus of voting. We leave an interaction analysis of the electoral rules and strategic position-taking for further research. When parties endogenously determine the set of competing candidates, the literature agrees that low-quality candidates adopt extreme positions to soften valence competition, while high-quality candidates adopt centrist positions (Groseclose, 2001; Aragones and Palfrey, 2002; Hollard and Rossignol, 2008; Hummel, 2010) even if valence is endogenous for the campaigning (Carrillo and Castanheira, 2008; Ashworth and de Mesquita, 2009).

We presume that *only two dimensions* emerge as relevant for describing the candidates. If only two dimensions matter, the non-conflicting valence dimension can end up either as (i) an irrelevant dimension (i.e., there is no tradeoff between quality and policy), (ii) a relevant and strong dimension (i.e., low-quality candidates will likely be ousted by high-quality candidates), or (iii) a relevant but weak dimension. Only the last option is non-trivial and will be explicitly considered.

With only two dimensions, only four types of candidates exist. Motivation for having only a few candidates draws from the large Duvergerian literature on the number of serious candidates as an increasing function of the district magnitude M. The idea is that small district magnitudes make some social divisions latent and not expressed electorally.

Most of the vast Duvergerian literature examines a two-party prediction for simple plurality (Duverger's Law). The early literature offers models with Duvergerian equilibria in which all votes for the second challenger vanish (Palfrey, 1989; Myerson and Weber 1993; Cox, 1994). Iaryczower and Mattozzi (2013) replicate Duverger's Law in the presence of campaigning efforts. Dellis (2013) confirms Duverger Law for any top-scoring rule in a deterministic setting and risk aversion, where a top-scoring rule (a rule with a unique top score) is defined such that it allows the voter to cast a different score for the first and second candidate on the ballot. An exception is Patty, Snyder and Ting (2009) who point to the presence of 'too many' electoral equilibria in multi-candidate elections if the candidates are purely vote-seeking and adopt policy positions

independently on the qualities of the candidates or parties. Persson and Tabellini (2001) identify two effects: The first effect of multi-party systems is in the coalitional bargaining stage. A party with ideological similarity to the proposer becomes cheaper to include in a coalition than an ideologically distant party, which would tend to claim large rents in the bargaining. The second effect is of diluted individual performance. For large coalitions and closed party-lists, the misconduct of individual incumbents is more difficult to monitor, detect, and punish in elections. In addition, there is a lower incentive for challengers to monitor and reveal an incumbent's underperformance in multi-party systems since revelation activity by one challenger generates an uncompensated positive externality for the other challengers (Charron, 2011).

strategically. Among the most recent papers, Fujiwara (2011) uses a regression discontinuity in Brazilian mayoral elections to show that third-place candidates are more likely to be deserted in races under the simple plurality rule than in runoff elections.

In the analysis of multiple-seat districts, Cox (1994; 1997) predicts $M+1$ viable parties and in the long run, $M+1$ competing parties. This $M+1$ result echoes results from all-pay auctions, where the number of contenders for a winner-take-all contest with M prizes is typically $M+1$ (Siegel, 2009). The caveat is Cox formally develops the $M+1$ rule for a single non-transferrable vote (a single positive vote for $M>1$), and also his main evidence is primarily through district-level results from British and Japanese elections that used single non-transferrable voting.

The lack of a formal analysis of districts with larger magnitudes is also understandable because with larger magnitudes, there is a greater role for coordination on the nation-wide level and non-trivial linkages between district-wide and nation-wide competition (Cox, 1999). Morelli (2004) develops a model, where for sufficiently asymmetric preferences across districts, the linkages revert Duvergerian predictions; hence, the policy outcome with a proportional rule is more moderate than the one with plurality. Recent evidence (e.g., Singer and Stephenson, 2009) is nevertheless supportive of the Duvergerian hypothesis; hence, maintaining a low number of serious candidates is a reasonable point of departure.

3 The binomial game with symmetry

3.1 Players

Consider two binary dimensions: policy is L or R, and quality is zero or one. Four generic types of candidates are admissible. We let the candidate list K involve one candidate per each generic type, $K=\{L_1,L_0,R_1,R_0\}$.

Each voter is either of two types, $t=L,R$. A group of voters is the *ex ante majority* if its expected size is larger than the expected size of the other group. A group is the *ex post majority* if its realized size is larger than the realized size of the other group. In this section, we build a type-symmetric binomial model where expected sizes are identical. Assume a large fixed number $n\in\mathbb{N}$ of voters. Purely for technical convenience, let $\lfloor\frac{n}{6}\rfloor=\frac{n}{6}$. In both population models, each voter's probability of being an L-type is drawn from an independent and identical Bernoulli distribution with parameter $\tau_L\in(0,1)$, where for type-symmetry, $\tau_L=\frac{1}{2}$. The number of L-voters is a random variable $x\in N$ on the support $N\equiv\{x\in\mathbb{N}:0\leqslant x\leqslant n\}$ with a binomial distribution $B(x;n;\frac{1}{2})$. In the main analysis, we simplify the notation to $B(x)$.

Voters learn their private types right before the elections and do not communicate their types to the other voters. Voters make an inference about the aggregate x by the posterior distribution functions, where the types' posterior probability distribution functions for L-type and R-types are $B_L(x;n;\tau_L)$ and $B_R(x;n;\tau_L)$. The difference of posteriors for L-voters and R-voters is evident from $B_L(x;n;p) = B(x-1;n-1;\tau_L)$ for $x \geqslant 1$ and $B_R(x;n;\tau_L) = B(x;n-1;\tau_L)$ for $x \leqslant n-1$.[9] In particular, see $\frac{b_L(x)}{b(x)} = 2\frac{x}{n}$ and $\frac{b_R(x)}{b(x)} = 2\frac{n-x}{n}$, which imply that the posteriors of each type are 'optimistic' in a sense that for type t, the events of being in the ex post majority (t-majority events) are now seen as being more likely, and the events of being in the ex post minority (t-minority events) are now seen as less likely. For example, L-voters expect any particular L-majority event to be more likely than R-voters, $b_L(x) > b_R(x)$ for $x > n-x$, and vice versa.

Each type $t = L, R$ is characterized by the utility function $u_t(c)$ over the elected candidate $c \in K$, where the valuation of any elected candidate is invariant to the valuation of another elected candidate. Two separable arguments in the utility function are policy and quality of the candidate. Types are symmetrically antagonistic over the policy, and $V > 1$ denotes the common relative salience of the policy dimension to the corruption dimension. The assumption that the policy dimension is more salient than the quality dimension (*ideological bias*, see Krishna and Morgan, 2011) can be interpreted such that voters and the competing parties are sufficiently polarized; hence, voters consider the non-valent issue to be of first-order importance. By normalizing the benefit from the worst candidate to zero, the voters' objective functions are:

$$u_L(L_1) = V+1 > u_L(L_0) = V > u_L(R_1) = 1 > u_L(R_0) = 0,$$
$$u_R(R_1) = V+1 > u_R(R_0) = V > u_R(L_1) = 1 > u_R(L_0) = 0.$$

3.2 Electoral rules and admissible ballots

We consider a sub-class of scoring rules that are characterized by the maximal number of positive votes V^+ and the maximal number of negative votes V^-, where votes to a single candidate cannot cumulate. In any scoring rule, each voter's ballot is a vector that specifies the number of points that the voter assigns to each candidate. The vectors of points of all voters are summed into a vector of scores, and for an M-seat district, the winning candidates are M candidates with the highest scores. Ties for the M-seat are broken neutrally; a winner of the last seat is chosen randomly among all candidates involved in the tie, each with equal probability. The reason is to make an electoral rule neutral to any other aspect but the ballot structure.[10]

[9] For completeness, $B_L(0;n;\tau_L) = 0$ and $B_R(n;n;\tau_L) = 1$.

[10] In contrast, Meyerson (1993) allows ties to be broken by a secondary voter's ranking. The extra ranking is a technically very useful concept but involves three disadvantages: (i) In reality, the secondary ranking is not available unless list-voting with preferential votes is introduced. (ii) Non-neutral tie-breaking rules will tend to promote candidates with high valence (e.g., high-quality candidates) because in the construction of the secondary ranking, voters will *not risk any policy loss*. The tradeoff between a policy loss and quality gain is, however, a crucial tradeoff

In our setting, each voter of type t submits a ballot, $v_t = (v_t^{L_1}, v_t^{L_0}, v_t^{R_1}, v_t^{R_0})$. Formally, the scoring rules that we admit have two characteristics: (i) Since the votes to a candidate cannot cumulate, the number of points that the voter gives to a candidate is 1, 0, or -1 (a positive vote, no vote, a negative vote): $v_t^c \in \{1, 0, -1\}$. (ii) There are at maximum $V^+ \in \mathbb{N}$ positive votes and at maximum $V^- \in \mathbb{N}$ negative votes on the ballot. (By the *type* of vote, we mean whether a particular vote is positive or negative.) Hence, the ballot can be truncated. The first characteristic admits $4^3 = 64$ ballots, but the second characteristic reduces the set of feasible ballots. For the simplest rules such as plurality, we have only 5 feasible ballots. For the 2+1 rule, we have exactly 33 feasible ballots.[11] The largest set of 62 feasible ballots is admitted by combined approval-disapproval voting.[12] Additionally, we will restrict ourselves to admissible (weakly undominated) ballots.

In our sub-class of scoring rules, a particular electoral rule is a triplet (M, V^+, V^-). We will be examining the following rules: For $M = 1$, consider plurality (the 1+0 rule), the mixed system (the 1+1 rule), and approval voting ($V^+ = \#K - 1$; for our quadruplet of candidates, the 3+0 rule). For $M = 2$, consider plurality-at-large (the 2+0 rule), the 2+1 rule, approval voting, and combined approval-disapproval voting ($V^+ = V^- = \#K - 1$; here the 3+3 rule).

In this section, we seek pure-strategy equilibria. A voting profile $\sigma(v,t)$ in pure strategies satisfies $\sigma(v,t) \in \{0,1\}$ for any (v,t). The set of feasible pure-strategy voting profiles is a Cartesian product of the sets of feasible ballots, and this set is determined by the electoral rule. Throughout the analysis, it will be often useful to alternatively describe the pure strategy ballot not as a vector of points given to candidates, but as a vector of 'uses' of feasible votes, hence a vector of $V^+ + V^-$ elements. Each element is either *active* (a vote is put on the ballot for some candidate $c \in K$) or *inactive* (a vote is put on the ballot, and the element is $\varnothing$). We assume that voting is costless, which does not imply that each voter necessarily actively assigns all feasible votes.[13] Taking the inactivity of votes into account, the cardinality of the set of feasible pure strategy profiles in our electoral situation is only 25 for the plurality rule, but amounts to 999 for the 2+1 rule and $3,844$ for CAV.

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of the electoral rules. If the aim is to see how the electoral rule itself affects the quality of serious candidates, the incentive to support high-quality candidates must be purely endogenous, and the electoral rule should be 'neutral' in the tradeoff. (iii) With the extra assumption in favor of high-quality candidates, it is likely that changes of the electoral rules will effectively have no difference. This will only stem from the fact that the tie-breaking rule will suppress the effects of the electoral rules.

[11] These are permutations of the complete ballot $(1,1,0,-1)$ and truncated ballots $(1,1,0,0)$, $(1,0,0,0)$, $(1,0,0,-1)$, and $(0,0,0,-1)$.

[12] Given that $V^+ = V^- = \#K - 1 = 3$, ballots $(1,1,1,1)$ and $(-1,-1,-1,-1)$ are not feasible.

[13] Weak dominance does not imply activity of all votes. For a small set of candidates such as in our case, a voter can be made strictly worse off by being forced to use all votes actively (c.f., Feddersen and Pesendorfer, 1996).

Let $S_c(x,\sigma)$ be the score of candidate $c \in K$ under voting profile σ and realization x. (In the proofs, we avoid arguments to save space.) By type-symmetry (see Assumption 1 below), the candidate's score is

$$S_c(x,\sigma) = xv_L^c + (n-x)v_R^c.$$

We use the scores to characterize the candidates' probabilities of being elected for a given x for voting profile $\sigma(v,t)$. For each candidate and each x, we use an indicator variable $I_c|x = 0,1$. A candidate's probability of being elected conditional on event x is $\Pr(I_c = 1|x)$, and his or her *seat probability* is $\sum_{k=0}^{n} \Pr(I_c = 1|x)b(x)$.

For a given x, exactly M candidates with the highest scores $S_c(k,\sigma)$ win M seats. Let the ordered candidates' scores be $S_1(x,\sigma) \geqslant S_2(x,\sigma) \geqslant S_3(x,\sigma) \geqslant S_4(x,\sigma)$. Then, we have $\Pr(I_c = 1|x) = 1$ if $S_c(x,\sigma) > S_M(x,\sigma)$, and $\Pr(I_c = 1|x) = 0$ if $S_c(x,\sigma) < S_M(x,\sigma)$. In the case of a tie, recall that all candidates are treated identically, and the seat is allocated randomly. Thus, if $S_c(x,\sigma) = S_M(x,\sigma)$, then $\Pr(I_c = 1|x) = \frac{1}{z}$, where z is the number of candidates who satisfy $S_c(x,\sigma) = S_M(x,\sigma)$.

3.3 Pivotal events and seriousness

For each voting profile σ, we identify the events (i.e., the sets of realizations x) in which an individual voter is decisive, called *pivotal events*. The gains and losses in the pivotal events will shape the voter's best response ballot. More precisely, consider the profile σ and suppose any unilateral deviation of a single voter, characterized by the profile σ'. *Pivotal events* for the pair of profiles (σ,σ') are all events where the vector of candidates' probabilities of being elected changes. Notice that pivotal events directly depend on the given pair of profiles and indirectly are rule-specific in the sense that the electoral rule determines which feasible profiles σ and also which alternative profiles σ' are feasible given σ. To avoid excessive notation, we leave the analysis of the pivotal events to each particular electoral rule.

At this stage, it is only valuable to see that for the scoring rules where $v_t^c \in \{-1,0,1\}$, a pivotal event is either a tie or a near tie for the M-th seat. A *tie* x is characterized by $S_M(x,\sigma) = S_{M+1}(x,\sigma)$. For any electoral rule we consider, each tie for the M-th seat is obviously a pivotal event for some feasible pair (σ,σ'). The remaining pivotal events are in *near ties*, where a necessary condition for a near tie is $S_M(x,\sigma) - S_{M+1}(x,\sigma) \in \{1,2,3,4\}$ or $S_M(x,\sigma) - S_{M-1}(x,\sigma) \in \{1,2,3,4\}$. The reason to account for the differences of at most four points from the score of M-th candidate $S_M(x,\sigma)$ is that in the class of rules using non-cumulated $+1$ points and -1 points, a single voter changes an individual candidate's score at maximum by two points (e.g., by adding a positive vote and withdrawing a negative point), and therefore, the relative scores of two candidates cannot be affected if the difference is by five points or more. Typically, however, the relevant near ties occur only for differences in scores by one or two points.

The set of pivotal events involves all sub-sets of the sets, including all singletons x. Henceforth, it is convenient to decompose the analysis of gains and losses into the analysis of singletons x. For any pivotal singleton x constructed from a pair of profiles (σ, σ'), there must be at least a pair of candidates whose probabilities of getting elected at x, $\Pr(I_c = 1|x)$, have changed. Any candidate whose probability of getting elected for a pivotal event x can change under some pair (σ, σ') is called *a serious candidate in event x*. A candidate is called *serious* if $x \in N$ exists such that a candidate is serious in event x.

With the above classification of candidates, we can describe each active vote on the ballot (i.e., a positive or negative point) either as a *serious* or *non-serious* vote. A serious vote is cast to a serious candidate. By definition, a serious vote affects the seat probability of a corresponding serious candidate. A non-serious vote is a vote cast to a non-serious candidate.

3.4 Equilibrium concept

Besides a focus on the pure-strategy equilibria of admissible ballots, Assumption 1 characterizes the relevant equilibria as symmetric.

Assumption 1 (Symmetry) *In a relevant equilibrium, ballots are characterized only by types and are type-symmetric.*

The assumption involves two symmetries: *Within-type symmetry* (homogeneity) states that voters of an *identical type* behave in the equilibrium identically; the ballot of any L-type is v_L and the ballot of any R-type is v_R. Unlike Poisson game where payoff-irrelevant type sub-divisions have zero effect on marginal probabilities for strategy profiles (Myerson, 1998), multinomial games are not invariant to payoff-irrelevant type subdivisions. Hence, we must directly assume that there is no device that would instruct the voters of the same type to differ in their ballots. *Across-type symmetry* states that the L-voter's ballot is type-symmetric to the R-voter's ballot; hence, $v_R = (v_L^{R_1}, v_L^{R_0}, v_L^{L_1}, v_L^{L_0})$. As a shortcut, we can henceforth represent each symmetric equilibrium voting profile only by the ballot v_L. The next assumption is to consider equilibria that arise only in large electorates.

Assumption 2 (Large binomial game) *An equilibrium for feasible population size n is relevant only if the equilibrium exists also for any feasible population size $n' > n$.*

Similar to Bouton et al. (2012), we additionally apply a sincere stability refinement. This refinement is a behaviorally relevant and technically useful perturbation in large voting games. A strict equilibrium is always sincerely stable because additional pivotal events that arise with very small numbers of sincere votes must tend to vanish in importance in a large electorate relative to the original pivotal events. Sincere stability matters only for weak equilibria that contain no pivotal events because both types submit highly correlated (in our case, often identical) ballots, and the set of winners is independent of the event.

Assumption 3 (Sincere stability) *A weak equilibrium is relevant only if $\bar{s} > 1$ exists such that any voting game perturbed by any $0 < s \leqslant \bar{s}$ of extra sincere voters contains this equilibrium.*

Finally, we describe our approach in identifying the equilibria in the binomial game:

1. We construct a set of symmetric pure-strategy voting profiles (Assumption 1). We eliminate profiles with those apparently inadmissible ballots[14] that involve a positive vote to the worst candidate ($v_L^{R_0} = v_R^{L_0} = 1$) or a negative vote to the best candidate ($v_L^{L_1} = v_R^{R_1} = -1$) in order to obtain the set of candidate strategy profiles $\mathscr{V}$.

2. For each profile $\sigma \in \mathscr{V}$, we derive the corresponding candidates' score functions $S_c(x, \sigma), c \in K$ and the function $S_M(x, \sigma)$.

3. For the score functions, we identify pivotal events that are either ties or near ties for the M-th seat. This is relatively straightforward given that the score functions are linear in a single variable x.

4. The pivotal events identify the sets of serious candidates and non-serious candidates.

5. We check whether the voting strategies are best responses. That is, for each type t, we consider all admissible unilateral deviations. Relevant deviations are such that a pair (c, x) exists where $\Pr(I_c = 1|x)$ changes.

6. For each relevant deviation, we calculate each voter's expected win gain (positive or negative) at each pivotal singleton x where the candidates' probabilities of being elected have changed. The total expected gain is then the weighted sum of the expected gains times the posterior probabilities of the pivotal singletons $b_t(x)$. A profile σ is an equilibrium only if each total expected gain is non-positive.

[14] An alternative ballot that would make the vote inactive would weakly dominate this ballot. For other ballots, admissibility can be comprehensively evaluated only by constructing score functions for all feasible profiles (v_L, v_R) and identifying all pivotal events from all realizations $x \in N$.

7. For all weak or strict equilibria in admissible strategies, we check whether the equilibrium is invariant to the size of the electorate n (Assumption 2), and we control for sincere stability (Assumption 3).

An electoral outcome $O(\sigma, x)$ is a function that yields the vector of the candidates' probabilities of being elected for every x given profile σ. When these probabilities are zero or one, we can alternatively speak of the set of elected candidates. When characterizing electoral outcomes generated in the relevant equilibria, we will be primarily checking the pivotal events x when an electoral rule assigns in the equilibrium a seat to a Condorcet winner with probability one and the events when a Condorcet loser gains a seat with a positive probability. With two types of voters, Condorcet winners and Condorcet losers are defined by the preferences of the ex post majority group of voters. For $x \leqslant \frac{n}{2}$, R_1 is the Condorcet winner and L_0 is the Condorcet loser. For $x \geqslant \frac{n}{2}$, L_1 is the Condorcet winner and R_0 is the Condorcet loser.

3.5 Single-member districts

Most of the pivotal voting analysis in past literature has focused on single-member districts. Here, we briefly illustrate the effects of enriching the plurality ballot by extra positive and negative votes in our electoral situation. We will observe that extra positive votes help to alleviate coordination problems, while extra negative votes are typically less effective because of generating too many serious candidates.

By Proposition 1, the plurality rule exhibits a classic coordination problem with multiple relevant equilibria: Either low-quality candidates $\{L_0, R_0\}$ compete against each other, or high-quality candidates $\{L_1, R_1\}$ compete against each other. Both equilibria share a unique pivotal tie $x = \frac{n}{2}$ with a pair of serious candidates; hence, there is a close race only if the populations of R-voters and L-voters are balanced. The other candidates are non-serious. The equilibrium best response is always to support actively own policy candidates. (For the proofs, always see the appendix.)

Proposition 1 (Plurality) *For plurality, there are two strict equilibria: (i) L-voters support L_1 and R-voters support R_1. (ii) L-voters support L_0 and R-voters support R_0.*

Both equilibria are strict; hence, the single positive vote is always active and serious. The first equilibrium is associated with a sincere ballot, $v_L = (1, 0, 0, 0)$. For any x, the Condorcet winner is elected and the Condorcet loser is not elected. The second equilibrium is associated with a non-sincere ballot, $v_L = (0, 1, 0, 0)$, but voting is sincere over the subset of serious candidates, $K' = \{L_0, R_0\}$. For any x, the Condorcet winner is a non-serious candidate, thus not elected.

The Condorcet loser is surely elected in a tie $x = \frac{n}{2}$. The first equilibrium Pareto-dominates the second equilibrium.

What is the effect of adding a negative vote to the ballot? In the mixed (1+1) rule, we observe that a negative vote increases the pool of serious candidates. More specifically, the voters tend to cast positive and negative votes for a pair of serious candidates. Negative points from one group then cancel out positive points from the opposing group. As a result, in a tie $x = \frac{n}{2}$, serious candidates have scores that equal zero. However, the other two candidates also have scores that equal zero; hence, they must also be considered serious.

With all candidates considered serious, strategic mixing becomes very likely. Mixing then implies that low-quality candidates win seats with a positive probability. Proposition 2 proves that the incentive to mix is absent only when the pair $\{L_1, R_1\}$ attracts both positive and negative votes, and ideological bias V is very low.

Proposition 2 (Mixed rule) *In the mixed rule, a relevant equilibrium exists if and only if $V \leqslant 2$. In the relevant equilibrium, L-voters cast positive votes to L_1 and negative votes to R_1, and R-voters cast positive votes to R_1 and negative votes to L_1.*

In a special case, the electoral outcome under profile $v_L = (1, 0, -1, 0)$ is identical to the outcome for the Pareto-superior equilibrium from plurality. Thus, we may conclude that the negative vote helps to eliminate the Pareto-inferior equilibrium but only in a small parametrical sub-space. In the remaining cases, once voters focus on reducing the chances of the opposing candidates, they cancel out each other's votes, and there is a window of opportunity for the weak and non-serious candidates with strategic mixing as a result.

What is the effect of adding a positive vote, which for truncated ballots amounts to approval voting? In our particular electoral situation, admissibility is sufficient to yield a unique favorable equilibrium under AV. The reason is the strategy $v_L = (0, 1, 0, 0)$ characterizing the Pareto-inferior equilibrium is weakly dominated by $v_L = (1, 1, 0, 0)$.

Proposition 3 (AV, $M = 1$) *For approval voting in single-member districts, L-voters approve L_1 and R-voters approve R_1.*

The unique equilibrium ballot under approval voting is identical to the Pareto-superior ballot in plurality, but in contrast to plurality, the equilibrium now involves two inactive votes. This may be considered as a disadvantage if we consider, instead of weak dominance, a selection criterion that requires all votes to be active unless inactivity brings a positive gain.

3.6 Two-member districts

We now proceed to the analysis of double-member districts. We will discuss both quality and representativeness of the candidates. In single-member districts, the ex post majority always wins the seat, and the only performance criterion is the incidence of high-quality candidates. For two-seat districts, the ex post majority also wins the first seat, but the second seat may fall to either of the groups. Thus, another natural criterion is how an electoral system protects minority interests. We will ask for which rule and under what conditions the second seat is allocated to a high-quality candidate of the minority group.

If only two positive votes can be cast, Proposition 4 finds a sincere and strict equilibrium with all votes being active. In the electoral outcome, the ex ante majority always wins both seats. The second seat is always for the inferior quality candidate.

Proposition 4 (Plurality-at-large) *For plurality-at-large, L-voters support $\{L_1, L_0\}$ and R-voters support $\{R_1, R_0\}$.*

Next, we consider the electoral rules with arbitrary numbers of votes, namely AV (positive votes) and CAV (positive and negative votes). For CAV, although the voters are given the option to discriminate in two dimensions by arbitrarily mixing positive and negative votes, they strategically 'overstate' their preferences and discriminate only by a single degree.[15] In the relevant equilibrium of both rules, electoral competition only reduces to the single policy dimension, exactly as in plurality-at-large.

Proposition 5 (AV and CAV, $M = 2$) *For approval and combined approval-disapproval voting in two-member districts, L-voters approve $\{L_1, L_0\}$ and R-voters approve $\{R_1, R_0\}$. In addition, for combined approval-disapproval voting, L-voters disapprove $\{R_1, R_0\}$ and R-voters disapprove $\{L_1, L_0\}$.*

Next, we consider the 2+1 rule. Proposition 6 identifies a unique, sincere, and strict equilibrium where all candidates are serious if $V \leqslant 3$. Interestingly, voters focus on punishing the electorally weaker candidate from the other camp instead of punishing the electorally stronger candidate. Like in plurality-at-large, the positive votes are cast along the more salient dimension (first-order discrimination). The difference is the negative vote is now cast along the less salient dimension (second-order discrimination).

[15] This 'overstating' dominance holds almost always in a deterministic setting (Felsenthal, 1989). In a large (stochastic) voting game that is solved by Myerson-Weber's ordering condition, Núñez and Laslier (2013) find that a class of evaluative voting rules yields identical equilibria like approval voting. Under evaluative voting with $m \in \mathbb{N}$ points, a voter can assign up to m points to each candidate. Approval voting is a special case of evaluative voting, where $m = 1$, and combined approval voting is a special case of $m = 2$.

Proposition 6 (2+1) *For the 2+1 rule, if and only if $V \leqslant 3$, a strict equilibrium exists where L-voters support $\{L_1, L_0\}$ and punish R_0, and R-voters support $\{R_1, R_0\}$ and punish L_0.*

The equilibrium survives if the expected gains in the pivotal events motivate the voters to cast a negative vote sincerely. There are two effects at play. The first effect concerns the *nominal gains in the pivotal events*. In the equilibrium, two pivotal ties $(x_A, x_B) = (\frac{n}{3}, \frac{2n}{3})$ arise. Consider L-voters: (i) If L-group is in the ex post majority ($x = x_B$), the weaker candidate L_0 competes with the stronger candidate R_1 for the second seat. The nominal gain is $V - 1$. (ii) If L-group is in the minority ($x = x_A$), the stronger candidate L_1 competes with the weaker candidate R_0 for the second seat. The nominal gain is $V + 1$; hence, this minority event is nominally more valuable.

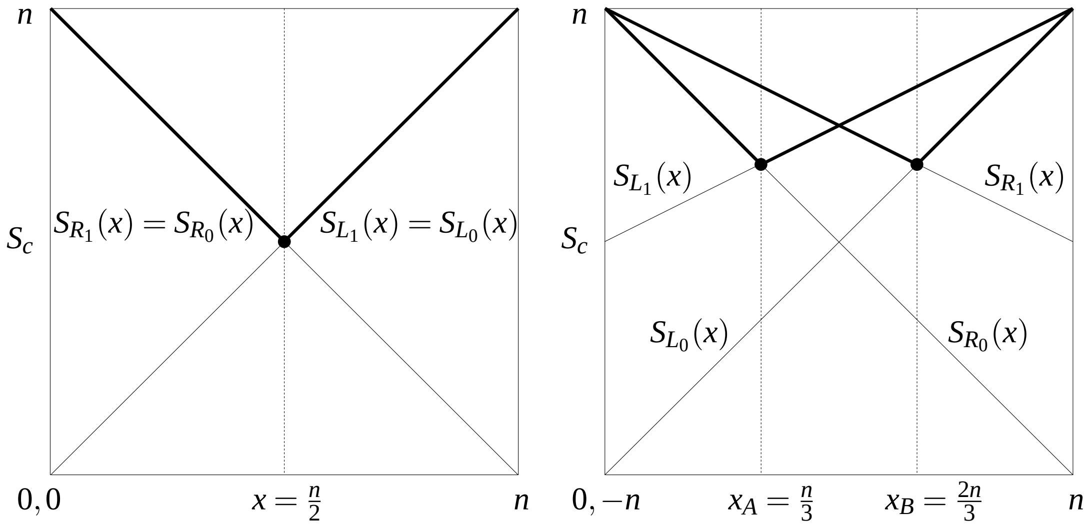
The second effect concerns the relative frequencies of pivotal events. We know that $\frac{b_L(x)}{b_L(n-x)} = \frac{n-x}{k}$; minority event x_A is seen by L-voters as more likely than majority event x_B, $b_L(x_B) = 2b_L(x_A)$. This reduces the appeal of sincere voting. Nevertheless, it can be easily proved that this effect is specific for the binomial game and can be suppressed in a symmetric finite Poisson game, where the environmental equivalence property makes both pivotal ties, x_A and x_B, equally likely.

The beneficial performance of the 2+1 rule can be attributed to the extended scope for a voter's discrimination associated with an additional negative vote. Once voters protect their primary policy interests by means of two positive votes, they can secure their secondary quality interests by an extra vote. A crucial difference is whether the extra vote is positive or negative. If the extra third vote is positive such as in approval voting, the restriction that votes of a certain type cannot cumulate for a single candidate binds. In contrast, this restriction of cumulation does not bind the use of an extra negative vote.

The existence of multiple ties is sustained because of the unequal number of non-cumulated positive and negative votes ($V^+ > V^-$). The unequal number generates a difference between positive and negative votes in the *substitutability (transferability) of votes across events*. While a negative vote can be transferred between candidates from the opposing group, a positive vote cannot be transferred because positive votes cannot cumulate. If positive votes could be freely transferred, the voter would deviate by shifting all positive and negative votes to a single most important pivotal event, and multiple ties would disappear. The inability to transfer the positive votes associated with an impossibility to cumulate votes is thus one of the necessary conditions for having two degrees of discrimination.

3.7 A comparison of the electoral outcomes

Only two electoral outcomes emerge out of the relevant strict profiles. In a *non-mixed electoral outcome* denoted O_n, the ex post majority always elects its first two candidates. This is associated



(a) O_n for plurality-at-large (b) O_m for the 2+1 rule

Figure 1: Electoral outcomes (elected candidates) and the candidates' scores

with plurality-at-large, AV and CAV. In a *mixed electoral outcome* generated by the 2+1 rule and denoted $O = O_m$, candidates from both groups are elected only if the differences in group sizes are not too large. Figure 1 illustrates the electoral outcomes, where O_n is depicted as the outcome of the plurality-at-large equilibrium.

The two outcomes identically allocate the first seat to the Condorcet winner (R_1 if $x < \frac{n}{2}$ and L_1 if $x > \frac{n}{2}$; for $x = \frac{n}{2}$, there is a tie). The effective comparison is only about the second seat. While both outcomes elect R_0 for the second seat for $x < \frac{n}{3}$ and L_0 for $x > \frac{2n}{3}$, the difference is the mixed outcome elects a high-quality minority candidate (L_1, respectively R_1) for $\frac{n}{3} < x < \frac{2n}{3}$ instead of a low-quality majority candidate (R_0, respectively L_0).

We assess quality and the representativeness of the electoral outcomes. In terms of quality, O_n delivers only a single high-quality candidate for all realizations except for $x = \frac{n}{2}$. In the special case of $x = \frac{n}{2}$, O_n even elects Condorcet losers with a positive probability. O_m involves two high-quality candidates for any $\frac{n}{3} < x < \frac{2n}{3}$ and never elects a Condorcet loser. Thus, O_m unambiguously ranks better in terms of quality. In terms of representation, a minority group never receives a seat under O_n; for O_m, it receives a seat for $\frac{n}{3} < x < \frac{2n}{3}$. The outcome that is more desirable depends on the weight assigned to the representation of the minority. One approach is to compare the outcomes in terms of utilitarian welfare. By this criterion, an ex post minority candidate is preferred if the ex post minority is not too small and if the common value (quality) of the minority candidate is much higher than the common value of the majority candidate.

For computational convenience, we compare the welfare of the two outcomes as an approximation at $n \to \infty$. Specifically, let $\phi := \frac{x}{n}$. For $n \to \infty$, $F(\phi)$ is the asymptotical distribution of the binomial distribution $B(x)$ for the size of voters normalized to unity, which is a normal distribution on the domain $\phi \in [0,1]$ with a mean $\frac{1}{2}$ and a standard deviation $\frac{1}{4}$.

For any outcome $O \in \{O_n, O_m\}$, the expected utilitarian welfare from the second seat is

$$W(O) \equiv \int_0^1 [\phi u_L(c_2(O,\phi)) + (1-\phi)u_R(c_2(O,\phi))] f(\phi)d\phi,$$

where $c_2(O,\phi)$ denotes the second elected candidate for a given share of L-voters ϕ under a given outcome O. To start with, we derive the socially optimal candidate. Given the absence of the first-seat candidate, we call the socially optimal candidate for the second seat the *second-best candidate*. For $\phi \leqslant \frac{1}{2}$, R_1 is the first-seat candidate, and

$$c_2^*(\phi) = \arg \max_{\{R_0,L_1,L_0\}} \phi u_L(c) + (1-\phi)u_R(c).$$

Trivially, since $u_t(L_1) > u_t(L_0)$ for both types, the second-best candidate for $\phi \leqslant \frac{1}{2}$ must be either R_0 or L_1 for $\phi \leqslant \frac{1}{2}$. The key inequality characterizing the second-best candidate is

$$\phi u_L(L_1) + (1-\phi)u_R(L_1) = 1 + \phi V \lesseqgtr (1-\phi)V = \phi u_L(R_0) + (1-\phi)u_R(R_0).$$

The equality rewrites into $\hat{\phi}(V+1) = (1-\hat{\phi})(V-1)$ and yields a threshold level $\hat{\phi} := \frac{V-1}{2V}$. The threshold defines the second-best efficient candidate as follows:

$$c_2^*(\phi) = \begin{cases} R_0 & \text{if } \phi \leqslant \hat{\phi} \\ L_1 & \text{if } \hat{\phi} \leqslant \phi \leqslant \frac{1}{2} \\ R_1 & \text{if } \frac{1}{2} \leqslant \phi \leqslant 1-\hat{\phi} \\ L_0 & \text{if } 1-\hat{\phi} \leqslant \phi. \end{cases}$$

The threshold has intuitive comparative statics: With an increasing relative importance of common value (decreasing V), the threshold $\hat{\phi}$ decreases. Thus, it is socially more important to establish a high-quality (but minority) candidate than a majority (but low-quality) candidate.

Proposition 7 proves that under a necessary condition for the existence of a mixed outcome (i.e., the existence of a strict equilibrium under the 2+1 rule), $\hat{\phi} \leqslant \frac{1}{3}$. Hence, the mixed outcome more frequently elects the second-best efficient candidate and welfare-dominates the non-mixed outcome.

Proposition 7 (Welfare) *If a relevant equilibrium exists for the 2+1 rule, then the mixed electoral outcome welfare-dominates the non-mixed electoral outcome, $W(O_m) \geqslant W(O_n)$.*

4 A large Poisson game with asymmetry

Are the results from the symmetric binomial setting robust to asymmetries? With asymmetries, pure strategy equilibria are unlikely, and mixed strategies must be considered. Also, symmetry in strategies must be relaxed. The complications associated with the identification of relevant ties in mixed profiles are minimized by adopting a large Poisson game. That is, in Section 4, we are seeking the properties of equilibria in a sequence of finite Poisson games, where $n \to \infty$. We follow an approach to analyze equilibria by the means of Myerson's (2000) magnitude and offset theorems as in Bouton and Castanheira (2012) and Bouton (2013).

4.1 The 2+1 rule

To begin with, we demonstrate that a sincere profile of the 2+1 rule ceases to exist under asymmetry. Let x_L and x_R be the numbers of sincere L-ballots and R-ballots, which are Poisson variables with means $n\tau_L$ and $n\tau_R$. The main difference between Poisson and binomial setting is that a particular tie is not characterized by a unique realization x, but by a set of realizations (x_L, x_R). To realize whether the tie is a relevant pivotal event even for $n \to \infty$, we must calculate the magnitude of the tie. Intuitively, the magnitude is a 'speed' at which the probability decreases towards zero. Only those events with the highest magnitudes (i.e., the lowest speed) are the relevant events.[16]

W.l.o.g., we consider that L-voters constitute an ex ante majority, $\tau_L > \frac{1}{2} > \tau_R = 1 - \tau_L$. Candidates' scores are $S_{L_1} = x_L \geqslant x_L - x_R = S_{L_0}$ and $S_{R_1} = x_R \geqslant x_R - x_L = S_{R_0}$. Second-rank pivot events are either two-candidate or four-candidate ties. The four-candidate pivot tie $(x_L = x_R = 0)$ has a magnitude of -1, which is the lowest feasible magnitude; as a consequence, this pivot will be irrelevant for a sufficiently large n.

What remains is a pair of two-candidate ties: Tie 1 ($S_{R_1} > S_{L_1} = S_{R_0}$; hence, $2x_L = x_R$) and Tie 2 ($S_{L_0} = S_{R_1} > S_{R_0}$; hence, $2x_R = x_L$). The unconstrained magnitude of Tie 1 is maximized at $x_L = n\sqrt[3]{\frac{\tau_L \tau_R^2}{4}}$ and equals

$$mag\ (piv^*_{L_1/R_0}) = 3\sqrt[3]{\frac{\tau_L \tau_R^2}{4}} - 1.$$

Given that $2x_L = x_R$ implies the constraint $x_R \geqslant x_L$ ($S_{R_1} > S_{L_1}$), the unconstrained magnitude is the effective magnitude for any τ_L. The unconstrained magnitude of Tie 2 is maximized at

[16] A formal treatment of the problem and tools for the calculation of the magnitudes are available in Appendix A1 in Bouton and Castanheira (2012).

$x_R = n\sqrt[3]{\frac{\tau_L^2\tau_R}{4}}$ and equals

$$mag\ (piv^*_{L_0/R_1}) = 3\sqrt[3]{\frac{\tau_L^2\tau_R}{4}} - 1.$$

Similarly, $2x_R = x_L$ implies the constraint $x_L \geqslant x_R$ ($S_{L_0} > S_{R_0}$); hence, the unconstrained magnitude is the effective magnitude for any $\tau_L > \frac{1}{2}$. Since this effective magnitude is larger than the effective magnitude of Tie 1 for $\tau > \frac{1}{2} > \tau_R$, only Tie 2 is a relevant pivot event according to magnitude theorem (Myerson, 2000). The L-voter thus disregards Tie 1 for any sufficiently large n and deviates by strategically casting a negative vote for the high-quality candidate R_1. In a large Poisson game, a sincere profile thus remains an equilibrium only in a knife-edge case in a symmetric district, where $\tau_L = \tau_R = \frac{1}{2}$, and $mag\ (piv_{L_0/R_1}) = mag\ (piv_{L_1/R_0})$.

In contrast, in the binomial game, a sincere profile may remain an equilibrium for small asymmetries if $V < 3$. The binomial setting is illustrative: The likelihood ratio of the two ties x_A and x_B matters as it influences each voter deciding which of the two events to address with a single negative vote. If t-group becomes ex ante majority, the ex-ante-majority event of t is increasingly more likely, and the ex-ante-minority event of t is decreasingly less likely. For ex-ante-minority voters, sincere voting remains the best response because the more valuable minority event is now also an increasingly more likely event. For ex-ante-majority voters, however, the more valuable minority event becomes very unlikely, and the voters consider using the negative vote rather in the more likely (albeit nominally less valuable) majority event.

We now identify a mixed equilibrium of a largely asymmetric district in a large Poisson game. The mixed-strategy equilibrium of the 2+1 rule features a sincere ballot by (ex ante minority) R-voters and a *symmetric* mix of sincere and strategic ballots by (ex ante majority) L-voters. Interestingly, the equilibrium structure is invariant to the degree of asymmetry in a large Poisson game if the asymmetry is sufficiently large.

We let $\alpha \in [0,1]$ be the probability that any L-voter adopts a *strategic* ballot. To keep notation as simple as possible, let x_A and x_B be the number of sincere and strategic L-ballots, which are Poisson variables with means $\tau_A := (1-\alpha)n\tau_L$ and $\tau_B := \alpha n\tau_L$. For convenience, we introduce shares $\chi_J = \frac{x_J}{n}$, where $J = A, B, R$. The ratios $\frac{\chi_J}{\tau_J}$ are known as *offset ratios*.

Candidates' scores are $S_{L_1} = x_A + x_B \geqslant x_A + x_B - x_R = S_{L_0}$ and $S_{R_1} = x_R - x_B \lesseqgtr x_R - x_A = S_{R_0}$. Again, we focus only upon two-candidate ties. In contrast to the sincere profile, we now have to account for four types of two-candidate ties. The reason is that L-voters have abandoned purely sincere voting, and therefore $S_{R_1} \gtreqless S_{R_0}$. The ambiguity of the sign inequality generates two extra second-rank ties: Tie 3 ($S_{L_1} \geqslant S_{L_0} = S_{R_0} \geqslant S_{R_1}$) and Tie 4 ($S_{R_0} \geqslant S_{L_1} = S_{R_1} \geqslant S_{L_0}$) in addition to Tie 1 and Tie 2 as analyzed above.

The main idea of Proposition 8 is a mixed strategy in a large Poisson game requires multiple relevant pivot ties (c.f., Bouton and Castanheira, 2012). By the magnitude theorem, pivot ties are relevant only if their magnitude is identical and the maximum out of all magnitudes. In our case, with symmetric mixing, Tie 2 and Tie 3 have equal magnitude, $mag\ (piv_{L_0/R_1}) = mag\ (piv_{L_0/R_0})$; hence, the ratio of their probabilities will be constant with $n \to \infty$.[17] Ties 1 and 4 will have a lower magnitude, and the voters will thus cast their votes to affect only pivots at Ties 2 and 3. This implies sincere voting from R-voter (to outvote L_0) and a mixed ballot from L-voter (to outvote both R_0 and R_1).

In the binominal game, the pivot tie is characterized by a unique x. In a Poisson game, the pivot tie is characterized by an equality. Namely, for our class of mixed-strategy profiles, each pivot tie is fully characterized by a triplet of binary indicators (d_A, d_B, d_R), where $d_J \in \{1,2\}$, such that

$$d_R \chi_R = d_A \chi_A + d_B \chi_B.$$

For Tie 1, $(d_A, d_B, d_R) = (2,1,1)$; for Tie 2, $(d_A, d_B, d_R) = (1,2,2)$; for Tie 3, $(d_A, d_B, d_R) = (2,1,2)$; and for Tie 4, $(d_A, d_B, d_R) = (1,2,1)$. Lemma 1 uses these indicators to express the implicit forms of the magnitudes of the pivotal events.[18]

Lemma 1 (Magnitudes) *The ballots which maximize unconstrained magnitude of a tie characterized by $d_R \chi_R = d_A \chi_A + d_B \chi_B$ can be written as a triplet of the offset ratios*

$$\left(\frac{\chi_A}{\tau_A}\right)^{3-d_A} = \left(\frac{\chi_B}{\tau_B}\right)^{3-d_B} = \left(\frac{\tau_R}{\chi_R}\right)^{3-d_R}.$$

The magnitude is unconstrained if $\chi_A \leqslant \hat{\chi}$ (Ties 1 and 2), respectively $\chi_A \geqslant \hat{\chi}$ (Ties 3 and 4), where

$$\hat{\chi}^{d_B - d_A} = \tau_A^{3-d_A} \tau_B^{d_B - 3}.$$

[17] To put exactly, a small ε_n is involved in mixing, where $\lim_{n \to \infty} \varepsilon_n = 0$.

[18] Closed-form solutions are extremely complex. We derive for illustration a solution for Tie 1. First, we use Lemma 1 and enter $\frac{\chi_A}{\tau_A} = \frac{\chi_B^2}{\tau_B^2}$ into $\frac{\chi_B}{\tau_B} \frac{\chi_R}{\tau_R} = \frac{\chi_B}{\tau_B} \frac{2\chi_A + \chi_B}{\tau_R} = 1$ to obtain a polynomial $2\tau_A \chi_B^3 + \tau_B^2 \chi_B^2 - \tau_B^3 (1 - \tau_A - \tau_B) = 0$. The magnitude-maximizing ballots are characterized by the root of the polynomial

$$\chi_B = \frac{\tau_B^4}{3\tau_A \sqrt[3]{4} \sqrt[3]{-108\tau_A^3\tau_B^3 - 108\tau_A^2\tau_B^4 + 108\tau_A^2\tau_B^3 + \sqrt{(-108\tau_A^3\tau_B^3 - 108\tau_A^2\tau_B^4 + 108\tau_A^2 b^3 - 2\tau_B^6)^2 - 4\tau_B^{12}} - 2\tau_B^6}}$$
$$+ \frac{\sqrt[3]{-108\tau_A^3\tau_B^3 - 108\tau_A^2\tau_B^4 + 108\tau_A^2\tau_B^3 + \sqrt{(-108\tau_A^3\tau_B^3 - 108\tau_A^2\tau_B^4 + 108\tau_A^2\tau_B^3 - 2\tau_B^6)^2 - 4\tau_B^{12}} - 2\tau_B^6}}{6\tau_A \sqrt[3]{2}} - \frac{\tau_B^2}{6\tau_A}.$$

The ballots which maximize constrained magnitude of the tie can be written as $(\chi_A, \chi_B, \chi_R) = (\chi, \chi, \frac{d_A+d_B}{d_R}\chi)$*, where*

$$\chi^{d_A+d_B+2d_R} = \tau_A^{d_R}\tau_B^{d_R}\left(\frac{d_R\tau_R}{d_A+d_B}\right)^{d_A+d_B}.$$

Our approach in the analysis of ties is to focus on how unconstrained magnitudes change in the parameters (α, τ_L). To start with, we calculate for each pivot tie parametrical cases when the respective unconstrained magnitude is maximal. This case occurs if all expected ballots in fact characterize the pivot tie, $n\tau_J = x_J$, or $\chi_J = \tau_J$ for $J = A, B, R$. Intuitively, with Poisson distributions drawn out of these parameters, the probability mass concentrates with $n \to \infty$ exactly in the pivot tie, and the pivot tie is the most likely event in a large Poisson game. Consequently, $\chi_A + \chi_B + \chi_R = \tau_A + \tau_B + \tau_R = 1$, and the maximal magnitude is $\chi_A + \chi_B + \chi_R - 1 = 0$.

We introduce the *magnitude-maximizing function* $\tau_L^{L_i/R_j}(\alpha)$ for a given tie $S_{L_i} = S_{R_j}$ characterized by (d_A, d_B, d_B). Each such function is determined by $d_R\tau_R = d_A\tau_A + d_B\tau_B$, or $d_R(1-\tau_L) = d_A(1-\alpha)\tau_L + \alpha\tau_L$:

$$\tau_L^{L_i/R_j}(\alpha) = \frac{d_R}{d_A + d_R + (d_B - d_A)\alpha}.$$

Lemma 2 exploits the magnitude-maximizing functions to characterize the sign of the offset ratios relative to one, $\text{sgn}(\frac{\chi_J}{\tau_J} - 1) = \text{sgn}(\chi_J - \tau_J)$ for $J = A, B, R$.

Lemma 2 (Offset ratios) *Consider a pivot tie* $S_{L_i} = S_{R_j}$*, where* $i, j = 0, 1$*. The ballots which maximize unconstrained magnitude satisfy*

$$\text{sgn}\left(\frac{\chi_A}{\tau_A} - 1\right) = \text{sgn}\left(\frac{\chi_B}{\tau_B} - 1\right) = -\,\text{sgn}\left(\frac{\chi_R}{\tau_R} - 1\right) = -\,\text{sgn}\left(\tau_L - \tau_L^{L_i/R_j}(\alpha)\right).$$

Lemma 2 will be exploited in two main ways. First, notice that for symmetric mixing $\alpha = \frac{1}{2}$, we have $\tau_L^{L_0/R_1} = \tau_L^{L_0/R_0} = \frac{4}{7}$. Thus, for $\tau_L \geqslant \frac{4}{7}$ we have $\tau_L \geqslant \max\{\tau_L^{L_0/R_1}, \tau_L^{L_0/R_0}\}$. By Lemma 2, this is a sufficient condition for Ties 2 and 3 to have unconstrained magnitudes.

The second specific use is for proof of stability. With Lemma 2, we demonstrate that in the equilibrium, if L-voters vote more sincerely (α decreases), then Tie 2 will have a higher magnitude than Tie 3, and L-voters will deviate by strategically outvoting R_1. Similarly, if L-voters vote more strategically (α increases), then Tie 3 will have a higher magnitude than Tie 2, and L-voters will deviate by sincerely outvoting R_0.

Proposition 8 (2+1, asymmetric district) *For* $\tau_L \geqslant \frac{4}{7}$*, a stable equilibrium exists where R-voters cast a sincere ballot* $v_R = (0, -1, 1, 1)$*, and L-voters mix a sincere ballot* $v_L = (1, 1, 0, -1)$ *and a strategic ballot* $v_L = (1, 1, -1, 0)$ *with probabilities* $(\frac{1}{2} + \varepsilon_n, \frac{1}{2} - \varepsilon_n)$*, where* $\varepsilon_n = \frac{1}{2n}\log\frac{V}{V-1}$*.*

4.2 AV, CAV, and 2+0

In a symmetric binomial game, all electoral rules except for the 2+1 rule generate a non-mixed electoral outcome. All players approve only candidates of their type and disapprove candidates of the other type (if negative votes are available). Our next result is that such pure policy competition is robust to the introduction of a large Poisson game and holds for any asymmetry and any valuation.

Proposition 9 (Non-mixed equilibrium) *For plurality-at-large, AV, and CAV, the approval of own-group candidates and the disapproval of other-group candidates establishes an equilibrium for any parameters.*

How does the mixed equilibrium outcome of the 2+1 rule rank relative to non-mixed outcome? Consider quality first. The non-mixed outcome (almost) always elects a single high-quality candidate. The mixed-outcome differs only in realizations where either a double-quality pair $\{L_1, R_1\}$ or a single-quality pair $\{L_1, R_0\}$ are elected. Given that $S_{L_1} \geqslant S_{L_0}$, no other pair can be elected. Since the probability of a double-quality pair $\{L_1, R_1\}$ is positive, the incidence of high-quality candidates is larger. Although the mixed negative vote of L-voters reduces the beneficial effect of the negative vote in terms of quality, the effect remains positive.

Also, ex post minorities are better represented. In the non-mixed outcome, an ex post minority candidate is (almost) never elected. In the mixed outcome, an ex post minority candidate is elected if any type-mixed pair are elected, i.e., under $\{L_1, R_1\}$ or $\{L_1, R_0\}$.

5 Conclusions

In this paper, we have identified the equilibrium voting outcomes for alternative scoring electoral rules in a stylized electoral situation with a single ideological dimension and four generic types of candidates. Two randomly sized groups of rational voters vote primarily to promote their preferred ideologies, and secondarily to support high-quality candidates. We have compared the outcomes in single-member and two-member districts for scoring rules that differ in the maximal numbers of positive and negative votes.

In our electoral situation, plurality-at-large almost always ends in a non-split outcome, where two candidates of the same group gain both seats. This prediction corresponds to 80% of outcomes of the recent elections into the assemblies of U.S. states that actually employ plurality-at-large in double-member districts. The large incidence of non-split outcomes is due to a large correlation of scores of party candidates.

In the class of scoring rules, the correlation can be weakened by adding more votes on the ballot. However, by adding an unrestricted number of votes (i.e., *approval voting* with an arbitrary number of positive votes or *combined approval-disapproval voting* with an arbitrary number of positive and negative votes), the plurality-at-large outcomes only replicate.

In contrast, the 2+1 rule yields an equilibrium with a large frequency of split outcomes; hence, it protects interests of the minority voting group relatively better than alternative two-seat electoral rules. For symmetric distributions, the 2+1 rule even motivates all voters to cast a negative vote only to the low-quality candidate from the opponent's group. Two positive votes are cast along the more salient ideological dimension, and a single negative vote is cast along the less salient quality dimension. Under group asymmetry, this behavior remains unchanged for minority voters while majority voters mix their negative votes.

Although the main purpose of the paper has been to compare the 2+1 electoral rule with plurality-at-large and the closest alternatives in double-member districts, a more general purpose of the paper is to understand how scoring electoral rules in multi-member districts affect the voters' tradeoffs over valent and conflicting political issues. Adding an extra negative vote has a different effect than adding an extra positive vote because the expected scores of serious candidates and non-serious candidates are affected differently and because of the restraint on cumulating multiple points to a single candidate.

We find that the simultaneous presence of negative and positive votes increases the voters' scope for discrimination. Nevertheless, an effectively discriminative mix of positive and negative votes must avoid two phenomena: (i) the underdog effect of an excessively large set of viable candidates and (ii) the incentive to overstate, which results in the serious candidates receiving only points $-1, 1$, and not points $-1, 0, 1$. The 2+1 rule avoids both by limiting the number of negative votes relative to the number of the positive votes.

Our electoral situation has a direct parallel to the widely studied divided majority situation. In both electoral situations, voters seek multiple objectives: (i) the coordination of own group (against the other group), (ii) quality-discrimination within own group, and (iii) quality-discrimination within the other group. Plurality cannot achieve all the objectives at the same time. Bouton and Castanheira (2012) show that approval voting effectively aggregates information and avoids coordination failure in a divided majority situation. The voters achieve the first and second objective by mixing *multiple ballots* in a mixed strategy. In our electoral situation, some voters achieve the first and third objective at once by a *single ballot* in a pure-strategy. In brief, the two-level discrimination involved in the 2+1 rule promotes multiple objectives by means of a single ballot.

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A Proofs

A.1 Proof of Proposition 1 (Plurality)

W.l.o.g., we examine L-voter's deviations in the set of admissible profiles:

- With all votes inactive, $v_L = (0,0,0,0)$, all candidates are serious for every x, and L-voter deviates by $v_L = (1,0,0,0)$.
- Vote for the 1st candidate, $v_L = (1,0,0,0)$: There is a unique pivot tie $x = \frac{n}{2}$. In all pivotal events, L_1 and R_1 compete for the seat. L_0 and R_0 are not serious candidates. L-voter's pivotal events are $\{\frac{n}{2}, \frac{n}{2}+1\}$, and R-voter's pivotal events are $\{\frac{n}{2}-1, \frac{n}{2}\}$. L-voter may only lose by any deviation; hence, this profile is an equilibrium.
- Vote for the 2nd candidate, $v_L = (0,1,0,0)$: There is a unique pivot tie $x = \frac{n}{2}$. In all pivotal events, L_0 and R_0 compete for the seat. L_1 and R_1 are not serious candidates. L-voter may only lose by deviation, hence this profile is also an equilibrium.
- Vote for the 3rd candidate, $v_L = (0,0,1,0)$: There is a unique pivot tie $x = \frac{n}{2}$. In all pivotal events, L_1 and R_1 compete for the seat. L-voter deviates by $v_L = (1,0,0,0)$. $\square$

A.2 Proof of Proposition 2 (Mixed)

Since the 1+1 rule contains weakly more votes of any type than the 1+0 rule, any deviation present in the 1+0 rule is also a deviation in the 1+1 rule. This principle will be used in other proofs as well. When using this principle, however, bear in mind that M-seat ties count; hence, M must be constant for the applicability of the principle.

- For any equilibrium profile from the 1+0 rule, L-voter deviates by using his or her negative vote. The negative vote changes the near tie $x = \frac{n}{2} - 1$ into a tie and wins tie $x = \frac{n}{2}$.
- Positive vote inactive, negative vote active: Two candidates A and B (those not receiving negative votes) are serious for every x. If L-voter deviates by casting a positive vote to the preferred candidate out of A, B, he or she wins all realizations $x = 1, \ldots, n$ (but obviously not the realization $x = 0$). This is a strict improvement for $x = 1, \ldots, n-1$. For $x = n$, there may be a loss if candidate D, who is serious in event n, is very valuable. However, for a sufficiently large n, the discrete loss in a single realization is always compensated for by the sum of discrete gains in $n-1$ realizations.

We are left with those profiles where all votes are active. We first rule out profiles where a pair of candidates A,B receives positive votes and a pair C,D receives negative votes: There is a unique pivot tie $x = \frac{n}{2}$. In all pivotal events, A and B compete for the seat. C and D are not serious candidates. L-voter deviates by transferring the negative vote to the worse of the candidates A and B.

Thus, only two candidates receive both positive and negative votes. Thus, these votes cancel out in tie $x = \frac{n}{2}$, where $S_c(x) = 0$ for any $c \in K$. In the tie, the expected payoff is $\frac{2V+2}{4} = \frac{V+1}{2}$.

- $v_L = (0,1,0,-1)$: L-voter deviates to $v_L = (1,0,0,-1)$. Thereby, L-voter wins tie $x = \frac{n}{2}$ with the 1st candidate L_1 (gain). Also, L-voter changes the win of L_0 at near tie $x = \frac{n}{2}+1$ into a tie between L_1 and L_0 (gain). The elected candidate for $x < \frac{n}{2}$ and $x > \frac{n}{1}+1$ is not changed. This is not an equilibrium.

- $v_L = (1,0,-1,0)$: L-voter considers deviating to $v_L = (0,1,-1,0)$ (positive vote). Thereby, L-voter replaces the tie of four candidates at $x = \frac{n}{2}$ with a win for the 2nd candidate L_0 (gain $V - \frac{V+1}{2} = \frac{V-1}{2} > 0$), but, at the same time, L-voter changes the win of L_1 at near tie $x = \frac{n}{2}+1$ into a tie between L_1 and L_0 (a loss of $-\frac{1}{2}$). The elected candidates for $x < \frac{n}{2}$ and $x > \frac{n}{1}+1$ are not changed. The deviation does not make L-voter worse off iff $b_L(\frac{n}{2})(\frac{V-1}{2}) - b_L(\frac{n}{2}+1)\frac{1}{2} \geqslant 0$. We use $b_L(\frac{n}{2}) = b_L(\frac{n}{2}+1)$; hence, the condition rewrites into $V \geqslant 2$.

  L-voter may consider deviation to $v_L = (1,0,0,-1)$ (negative vote). The only effect is that R_1 wins $x = \frac{n}{2}$ with a loss $1 - \frac{V+1}{2} = \frac{1-V}{2} < 0$.

  Also, L-voter may consider a deviation to $v_L = (0,1,0,-1)$ (positive and negative vote). At $x = \frac{n}{2}$, there is now a tie of R_1 and L_0 with exactly zero gain. At $x = \frac{n}{2}+1$, there is now a tie of R_1 and L_1, which means a loss of $-\frac{V}{2} < 0$. Finally, it is easy to see that there is no better deviation than the one considered up to now. □

A.3 Proof of Proposition 3 (AV, $M = 1$)

First of all, we consider equilibrium profiles from the 1+0 rule: To rule out the Pareto-inferior profile, we use ballot $v_L = (0,1,0,0)$ is weakly dominated by ballot $v_L = (1,1,0,0)$. For the Pareto-superior profile, only two candidates are serious. L-voter supports the better of the two serious candidates. Thus, L-voter cannot add an extra positive vote to improve his or her payoff. Also, the ballot $v_L = (1,0,0,0)$ is not weakly dominated by any other ballot since in alternative profiles, there might be a tie between the 1st and 2nd (or 3rd) candidates, and casting the additional positive vote may imply a utility loss in the tie. The Pareto-superior profile remains the equilibrium profile.

The next set of profiles involve two positive votes:

- Votes for the 1st and 3rd candidates, $v_L = (1,0,1,0)$: In any x, there is a tie between L_1 and R_1. L-voter deviates by withdrawing the positive vote for R_1.

- Votes for two different pairs of candidates: In any $x \neq \frac{n}{2}$, there is a tie within the pair of preferred candidates. Suppose L-voter prefers candidates A and B, and $u_L(A) > u_L(B)$. Then, L-voter deviates by withdrawing the positive vote for B. This implies a gain for $x = \frac{n}{2}+1,\ldots,n$ and a potential loss at $x = \frac{n}{2}$. For a sufficiently large n, the discrete loss in a single realization is always compensated for by discrete gains in $\frac{n}{2}-1$ realizations.

Finally, consider all three votes to be cast, $v_L = (1,1,1,0)$. Then, the configuration is identical to the 1+1 rule with only one negative vote active. Two candidates A and B (those not receiving negative votes) are serious for every x. If L-voter deviates by casting a positive vote to the preferred candidate out of A,B, he or she wins all realizations $x = 1,\ldots,n$ (but obviously not the realization $x = 0$). This is a strict improvement for $x = 1,\ldots,n-1$. For $x = n$, there may be a loss if candidate D, who is serious in event n, is very valuable. However, for a sufficiently large n, the discrete loss in a single realization is always compensated for by the sum of discrete gains in $n-1$ realizations. □

A.4 Proof of Proposition 4 (Plurality-at-lage)

W.l.o.g., we examine the deviations of L-voter in the set of admissible ballots. We prove that all votes must be active:

- With all votes inactive, $v_L = (0,0,0,0)$, all candidates are serious for every x, and L-voter deviates by $v_L = (1,1,0,0)$.

- Vote for a single candidate A. For L-voter, there is only one relevant pivot tie $x = n$ of candidates B, C, and D. (Recall that a tie at $x = 0$ is not relevant for L-voter since the event $x = 0$ involves only a set of R-voters.) L-voter deviates by casting the extra positive vote for the best out of the candidates B, C, and D.

Three pairs of candidates may receive two positive votes:

- Vote for the 1st and 2nd candidates, $v_L = (1,1,0,0)$: There is a unique pivot tie $x = \frac{n}{2}$, where all four candidates compete for the seat. At the tie, any deviation makes L-voter strictly worse off. At near tie $x = \frac{n}{2}+1$, a transfer of a positive vote from own candidate

(L_1 or L_0) to any other candidate (R_1 or R_0) induces a tie, and this makes L-voter strictly worse off. Thus, this is a strict equilibrium for any n. As a sincere equilibrium, the profile is sincerely stable.

- Vote for the 1st and 3rd candidates, $v_L = (1,0,1,0)$: For any x, $S_{L_1}(x) = S_{R_1}(x) = n > 0 = S_{L_0}(x) = S_{R_0}(x)$. There is no pivotal event, and all candidates are non-serious. This is a weak equilibrium. We prove that it is not sincerely stable. Consider L-voter who expects x_L of sincere L-voters to vote $v_L = (1,1,0,0)$, x_R of sincere R-voters to vote $v_R = (0,0,1,1)$, and x_S of strategic voters to vote $v_S = (1,0,1,0)$ independently on their ideological type. Scores satisfy $S_{L_1} = x_L + x_S \geqslant x_L = S_{L_0}$, and $S_{R_1} = x_R + x_S \geqslant x_R = S_{R_0}$. Pivotal events arise under either of three cases:

  - $x_S = 0$. There are two pivotal events: (i) The pivotal tie $x_L = x_R$ is characterized by $S_{L_1} = S_{L_0} = S_{R_1} = S_{R_0}$. (ii) The pivotal near-tie $x_L = x_R - 1$ is characterized by $S_{R_1} = S_{R_0} > S_{L_1} = S_{L_0}$. Conditional on each of the two events, the best response is sincere ballot $v_L = (1,1,0,0)$.
  - $x_S = 1$: There are three pivotal events: (i) The pivotal tie $x_L = x_R + 1$ is characterized by $S_{L_1} > S_{L_0} = S_{R_1} > S_{R_0}$. (ii) The pivotal near-tie $x_L = x_R$ is characterized by $S_{L_1} = S_{R_1} > S_{L_0} = S_{R_0}$. (iii) The other pivotal tie $x_L = x_R - 1$ is characterized by $S_{R_1} > S_{R_0} = S_{L_1} > S_{L_0}$. Conditional on any of the three events, the respective best response set involves sincere ballot $v_L = (1,1,0,0)$.
  - $x_S \geqslant 2$: There are two pivotal events known from the previous case: (i) The pivotal tie $x_L = x_R + x_S$ is characterized by $S_{L_1} > S_{L_0} = S_{R_1} > S_{R_0}$. (ii) The other pivotal tie $x_L = x_R - 1$ is characterized by $S_{R_1} > S_{R_0} = S_{L_1} > S_{L_0}$. The best response is sincere ballot $v_L = (1,1,0,0)$.

  To sum up, the sincere ballot is the best response in any pivotal event; hence, it constitutes a best response without assessing likelihood ratios of the probabilities. Therefore, the profile is not sincerely stable.

- Vote for the 2nd and 3rd candidates, $v_L = (0,1,1,0)$: There is a unique pivot tie $x = \frac{n}{2}$, where all four candidates compete for the seat. At the tie, L-voter becomes better off by transferring a positive vote from the 3rd candidate R_1 to the 1st candidate L_1. □

A.5 Proof of Proposition 5 (AV and CAV, $M = 2$)

A.5.1 AV (3+0), $M = 2$

Since the 3+0 rule contains weakly more votes of any type than the 2+0 rule, any deviation present in the 2+0 rule is also a deviation in the 3+0 rule. (An identical sincere stability refinement applies here because the ballot $v_L = (1,1,0,0)$ can be considered sincere both in 2+0 and 3+0.) We investigate whether the profile from the 2+0 rule remains in the equilibrium for the 3+0 rule:

- Votes for the 1st and 2nd candidates, $v_L = (1,1,0,0)$: There is a unique pivot tie $x = \frac{n}{2}$, where all four candidates compete for the seat, and the expected payoff from both seats is $V+1$. An extension of the strategy set by approval voting means that at the tie, L-voter now considers a deviation to $v_L = (1,1,1,0)$. This wins a single seat for R_1 at $x = \frac{n}{2}$, and the payoff from the first seat is 1. There is a tie for the second seat for the remaining candidates, and the expected payoff from the second seat is $\frac{2V+1}{3}$. This deviation makes L-voter worse off because $1 + \frac{2V+1}{3} < V+1$ is equivalent to $1 < V$. At the same time, this deviation has no effect on the probabilities of being elected at any $x \neq \frac{n}{2}$. Thus, this profile remains as an equilibrium.

Additionally, consider all three votes to be cast, $v_L = (1,1,1,0)$. In contrast to $M = 1$, ties for the second seat are only for the extreme realizations $x \in \{0,n\}$. Consider now L's deviation to $v_L = (1,1,0,0)$ (a vote for the 3rd candidate is withdrawn). We have effects at three realizations:

- In tie $x = n$, L_1 and L_0 win the two seats, and the payoff from the two candidates is $2V+1$. The expected payoff without a deviation was $\frac{4}{3}(V+1)$; hence, there is a gain $2V+1-\frac{4}{3}(V+1) = \frac{2V-1}{3} > 0$.

- In the near tie $x = n-1$, there is now a tie over the second seat between R_1 and L_0; hence, the expected payoff is $V+1+\frac{V+1}{2}$. The expected payoff without any deviation was $V+2$; hence, there is a gain $\frac{V+1}{2} - 1 = \frac{V-1}{2} > 0$.

- In the near tie $x = 1$, L_1 gains the first seat, and R_1 and R_0 now compete for the second seat. The payoff is $V+1+\frac{1}{2}$. The expected payoff without any deviation was $V+2$; hence, there is a loss $-\frac{1}{2} < 0$.

Since $b_L(1) = b_L(n)$ and $b_L(n-1) = (n-1)b_L(1)$, the deviation makes L-voter better off if

$$b(1)\left(\frac{2V-1}{3} - \frac{1}{2}\right) + (n-1)b(1)\left(\frac{V-1}{2}\right) \geqslant 0.$$

The condition rewrites into $V \geqslant \frac{3n+2}{3n+1}$. For any sufficiently large $n' > \frac{2-V}{3(V-1)}$, we have $V > \frac{3n+2}{3n+1}$. As a consequence, this profile is not an equilibrium for n'. The profile violates largeness Assumption 2.

A.5.2 CAV (3+3), $M = 2$

The 3+3 rule admits all ballots under 2+1 and 3+0 rules. Formally, the set of profiles Ψ^{3+3} satisfies $\Psi^{3+3} \supset \Psi^{2+1}$ and $\Psi^{3+3} \supset \Psi^{3+0}$. Therefore, we first verify the stability of the equilibria identified for these rules. For approval voting (3+0), we rule out all profiles on the grounds of inadmissibility; by admissibility, the worst candidate must receive a negative vote. For profiles identified by the 2+1 rule:

- $v_L = (1,0,1,-1)$ (a weak equilibrium for 2+1): No event is serious; hence, this remains a weak equilibrium. In Proof to Proposition 6, we have found that the original profile is dominated by the ballot $v_L = (1,1,0,-1)$, which also holds in this electoral rule because $\Psi^{3+3} \supset \Psi^{2+1}$. Thus, this profile is not sincerely stable.

- $v_L = (1,1,0,-1)$ (a strict equilibrium for 2+1): Each voter adds an extra negative vote to the high-quality candidate of the other group in order to change the winner in her less valuable pivotal event.

We examine all extra admissible profiles relative to the 2+1 rule. We begin with those that admit two or three negative votes, which is not feasible under 2+1:

- $v_L = (1,-1,-1,-1)$: For L-voter, consider tie $x = n$. By changing the score for L_0 into $v_L^{L_0} = 1$, L_0 wins this tie against R_1 and R_0, and no other effect takes place.

- $v_L = (1,1,-1,-1)$: There is a unique pivot tie $x = \frac{n}{2}$, where all four candidates compete for the seat. In a tie, any deviation makes L-voter *strictly* worse off. In a near tie $x = \frac{n}{2}+1$, any decrease in the points of own candidates (L_1 or L_0) and/or any increase in the points of the other candidates (R_1 or R_0) cannot make L-voter better. Thus, this is a strict equilibrium.

- $v_L = (1,-1,1,-1)$: There is no pivotal event since $S_{L_1}(x) = S_{R_1}(x) = n > -n = S_{L_0}(x) = S_{R_0}(x)$. Hence, this is a weak equilibrium. To examine sincere stability, note that $S_{L_1} = x_L + s_L + s_R - x_R \geqslant x_L - s_L - s_R - x_R = S_{L_0}$ and $S_{R_1} = x_R + s_L + s_R - x_L \geqslant x_R - s_L - s_R - x_R = S_{L_0}$, where (x_L, x_R) is for the numbers of sincere L-voters and R-voters, and (s_L, s_R) is for the numbers of strategic L-voters and R-voters.

  - $s_L + s_R = 0$. All candidates are serious and a sincere ballot is in the best response.

– $s_L + s_R \geqslant 1$. We have $S_{L_1} \geqslant S_{L_0} + 2$ and $S_{R_1} \geqslant S_{R_0} + 2$. There is a tie between the pair $\{L_1, L_0\}$ only if L-voter supports L_0 and punishes L_1, but punishing L_1 is not admissible. There is a tie between the pair $\{R_1, R_0\}$ only if L-voter supports R_0 and punishes R_1, but supporting R_0 is not admissible. Therefore, the relevant pivotal events are for pairs $\{L_1, R_0\}$ and $\{L_0, R_1\}$, where the best response is a sincere ballot, namely the support of $\{L_1, L_0\}$ and the punishment of $\{R_1, R_0\}$.

- $v_L = (1, 0, -1, -1)$: This profile is similar to the strict profile under the 2+1 rule. There are two ties, $x \in \{\frac{n}{3}, \frac{2n}{3}\}$, each with a pair of serious candidates $\{R_1, L_0\}$ and $\{R_0, L_1\}$. Each voter deviates by adding an extra positive vote to the low-quality candidate of own group to change the winner in her more valuable pivotal event.

- $v_L = (1, -1, 0, -1)$: There is no pivotal event since $S_{L_1}(x) = S_{R_1}(x) \geqslant 0 > -n = S_{L_0}(x) = S_{R_0}(x)$. Hence, this is a weak equilibrium. We examine sincere stability. This ballot involves the punishment of L_0 which requires a pivotal event exists where L_0 competes with L_1. Such an event is characterized by $S_{L_0} = x_L - x_R - s_L - s_R \geqslant x_L - x_R + s_L = S_{L_1}$. This is satisfied only for $s_L = s_R = 0$, but then all candidates are serious in a tie, and the best response is to support both $\{L_1, L_0\}$. Thus, the profile is not sincerely stable.

Finally, we examine the profiles that involve negative votes (unlike 3+0) and have more than two positive votes (unlike 2+1). This yields a single admissible profile:

- $v_L = (1, 1, 1, -1)$: L-voter deviates to $v_L = (1, 1, 0, -1)$. At tie $x = n$, there is a gain of having $\{L_1, L_0\}$ among the winners instead of $\{L_1, L_0, R_1\}$. In near ties $x = n - 1$ and $x = 1$, there are no effects on the sets of winning candidates. □

A.6 Proof of Proposition 6 (2+1)

The 2+1 rule admits all ballots under the 2+0 rule. Thus, any deviation present in the set of profiles for 2+0 rule, Ψ^{2+0}, is also a deviation in the set of profiles for the 2+1 rule, $\Psi^{2+1} \supset \Psi^{2+0}$. We first investigate whether the two equilibrium profiles from the 2+0 rule remain in the equilibrium for the 3+0 rule: By weak dominance, any admissible profile contains a negative vote. In this particular case, ballot $v_L = (1, 1, 0, 0)$ is weakly dominated by $v_L = (1, 1, 0, -1)$, and ballot $v_L = (1, 0, 1, 0)$ is weakly dominated by $v_L = (1, 0, 1, -1)$.

Consider now only a single negative vote being active. All such profiles involve pivotal ties for any x. All candidates are serious. L-voter deviates by casting positive votes for the 1st and 2nd candidates. The main reason for the improvement is for any x, a positive vote for the 2nd candidate cannot reduce the probability of the 1st candidate being elected, given that also the 1st candidate now obtains a positive vote.

Consider a positive and a negative vote being active. We exploit across-type symmetry to obtain only the following: Suppose L-voter supports A and punishes C.

- R-voter supports B and punishes D, where $\{A,B,C,D\}=K$. There are ties at extreme realizations, $x \in \{0,n\}$. For $x=0$, A competes with C. For $x=n$, B competes with D. L-voter can change the tie $x=n$ by giving an extra positive vote to the better of $\{B,D\}$. This deviation affects only the realization $x=n$.

- R-voter supports C and punishes A, and $(A,C)=(L_1,R_1)$: $v_L=(1,0,-1,0)$. There are pivotal ties for any x. For any x, L_0 is always a serious candidate. L-voter deviates by adding a positive vote to L_0: $v_L=(1,1,-1,0)$. For any $x \neq \frac{n}{2}$, L_0 now wins a seat in competition with R_0. For $x=\frac{n}{2}$, L_0 now wins the first seat, and the other three candidates compete for the second seat. This is also a strict improvement because $V+\frac{V+2}{3}>\frac{V+1}{2}$.

- R-voter supports C and punishes A, and $(A,C)=(L_0,R_0)$: $v_L=(0,1,0,-1)$. There are pivotal ties for any x, where L_1 is always a serious candidate. L-voter deviates by adding a positive vote to L_1; hence, $v_L=(1,1,0,-1)$. This is clearly an improvement for any x.

The remaining profiles are for all three votes being active. Trivially, we eliminate profiles where a positive and negative vote from one voter is for the same candidate because this would be the equivalent of casting no vote for the candidate and using only a single active negative vote. (These ballots have been eliminated under Ψ^{2+0}.) We are left with five profiles.

- Ballot $v_L=(1,-1,1,0)$. There is no pivotal event, and all candidates are non-serious. This is a weak equilibrium but not in admissible strategies.

- Ballot $v_L=(1,0,1,-1)$. There is no pivotal event, and all candidates are non-serious. This is a weak equilibrium in admissible strategies, but we will prove that it is not sincerely stable. Let (s_L,s_R) be the extra strategic L-voters and R-voters, where $s_L+s_R>0$. Then, scores are $S_{L_1}=n+s_L>s_L-(n-x)-s_R=S_{L_0}$, and $S_{R_1}=n+s_R>s_R-x-s_L=S_{R_0}$.

  Pivotal events involve a tie or near tie $S_{L_1}=S_{R_0}$ or $S_{R_1}=S_{L_0}$. L-voter thus never supports the third candidate R_1 because R_1 is serious only in events where also L_0 is serious. In other words, the motivation to support the third candidate exists only if the support of third candidate R_1 reduces the seat probability of the fourth candidate R_0. Yet this is impossible given that R_1 is not competing with R_0, $S_{R_1}>S_{R_0}$.

- Ballot $v_L = (1,1,0,-1)$. The vector of score functions is $(x, 2x-n, n-x, n-2x)$. There are two ties: $x_A = \frac{n}{3}$ and $x_B = \frac{n}{3}$. In tie x_A, L_1 and R_0 are serious. In tie x_B, L_0 and R_1 are serious. L-voter cannot reallocate any positive vote to gain in any x. The negative vote can be reallocated from R_0 to R_1. Then, R_0 wins a seat at x_A against L_1 instead of a tie (a loss $\frac{-(V+1)}{2} < 0$), and R_1 loses a seat at x_B against L_0 instead of a tie (a gain $\frac{V-1}{2} > 0$). Now, we use $b_L(x_B) = 2b_L(x_A)$. The expected gain is negative if and only if

  $$b_L(x_A)\left(-\frac{V+1}{2} + V - 1\right) < 0,$$

  which is equivalent to $V < 3$. Under this condition, the sincere profile is a strict equilibrium. For $V = 3$, the sincere profile is a weak (and sincerely stable) equilibrium.

- Ballot $v_L = (1,1,-1,0)$. The vector of score functions is $(2x-n, x, n-2x, n-x)$. There are two ties: $x_A = \frac{n}{3}$ and $x_B = \frac{2n}{3}$. In tie x_A, L_0 and R_1 are serious. In tie x_B, L_1 and R_0 are serious. L-voter cannot transfer any positive vote to realize gains for any x. The negative vote can be transferred from R_1 to R_0. Then, R_1 wins a seat at x_A against L_0 instead of a tie (a loss $\frac{-(V-1)}{2} < 0$), and R_0 loses a seat at x_B against L_1 instead of a tie (a gain $\frac{V+1}{2} > 0$). Again, we use $b_L(x_B) = 2b_L(x_A)$. The expected gain of a deviation is always positive,

  $$b(x_A)\left(-\frac{V-1}{2} + V + 1\right) = \frac{b(x_A)}{2}(V+3) > 0.$$

- Ballot $v_L = (0,1,1,-1)$. The vector of score functions is $(x, n-2x, n-x, 2x-n)$. There are two ties, $x_A = \frac{n}{3}$ and $x_B = \frac{n}{3}$. In tie x_A, L_0 and L_1 are serious. In tie x_B, R_1 and R_0 are serious. L-voter deviates by transferring a positive vote from L_0 to L_1. Then, L_1 wins a seat at x_A against L_0 instead of a tie (a gain $\frac{1}{2} > 0$). $\square$

A.7 Proof of Proposition 7 (Welfare)

By Proposition 6, the sufficient and necessary condition for a strict equilibrium under the 2+1 rule is $V \leqslant 3$. For $V \leqslant 3$, $\hat{\phi} \leqslant \frac{1}{3}$. Since $\hat{\phi} \leqslant \frac{1}{3}$, both mixed and non-mixed outcomes disproportionately favor R_0 to L_1. However, the distortion of the mixed outcome O_m occurs in the interval $\phi \in [\hat{\phi}, \frac{1}{3}]$ which is a proper subinterval of $\phi \in [\hat{\phi}, \frac{1}{2}]$ where the non-mixed outcome O_n distorts. Hence, under $V \leqslant 3$, $W(O_m) > W(O_n)$. $\square$

A.8 Proof of Lemma 1 (Magnitudes)

The first step in the maximization is to fix x_R and obtain $\left(\frac{\chi_A}{\tau_A}\right)^{3-d_A} = \left(\frac{\chi_B}{\tau_B}\right)^{3-d_B}$. The second step is to enter the equality back into the maximized magnitude, and by optimization receive

$\left(\frac{\chi_B}{\tau_B}\right)^{3-d_B} = \left(\frac{\tau_R}{\chi_R}\right)^{3-d_R}$.

We have to check if these ballots comply with the constraints. The only constraint that must be evaluated is $S_{R_0} \lesseqqgtr S_{R_1}$, or equivalently $\chi_A \lesseqqgtr \chi_B$. Specifically, Ties 1 and 2 require $\chi_A \geq \chi_B$, and Ties 3 and 4 require $\chi_A \leqslant \chi_B$. Using $\left(\frac{\chi_A}{\tau_A}\right)^{3-d_A} = \left(\frac{\chi_B}{\tau_B}\right)^{3-d_B}$, the inequality $\chi_A \geqslant \chi_B$ is equivalent to $\chi_A \leqslant \sqrt[d_B-d_A]{\tau_A^{3-d_A}\tau_B^{d_B-3}}$.

The constrained magnitude is maximized if $\sum_{J=A,B,R}\chi_J \log\frac{\chi_J}{\tau_J} = 0$ such that the single constraint binds, $\chi := \chi_A = \chi_B$. From the tie-characterizing condition, we express $\chi_R = \frac{d_A+d_B}{d_R}\chi$. By imposing these constraints, we have the maximization problem of a single variable, $\log\frac{\chi^2}{\tau_A\tau_B} + \frac{d_A+d_B}{d_R}\log\frac{\frac{d_A+d_B}{d_R}\chi}{\tau_R} = 0$ or $\frac{\chi^2}{\tau_A\tau_B}\left(\frac{\chi}{\tau_R}\right)^{\frac{d_A+d_B}{d_R}} = \left(\frac{d_R}{d_A+d_B}\right)^{\frac{d_A+d_B}{d_R}}$; hence,

$$\chi = \sqrt[d_A+d_B+2d_R]{\tau_A^{d_R}\tau_B^{d_R}\left(\frac{d_R\tau_R}{d_A+d_B}\right)^{d_A+d_B}}. \square$$

A.9 Proof of Lemma 2 (Offset ratios)

From Lemma 1, we observe that the unconstrained-magnitude-maximizing ballots satisfy $\text{sgn}(\chi_A - \tau_A) = \text{sgn}(\chi_B - \tau_B) = -\text{sgn}(\chi_R - \tau_R)$ or equivalently

$$\text{sgn}[d_A(\chi_A - \tau_A)] = \text{sgn}[d_B(\chi_B - \tau_B)] = \text{sgn}[d_A(\chi_A - \tau_A) + d_B(\chi_B - \tau_B)] = -\text{sgn}[d_R(\chi_R - \tau_R)].$$

First, consider a lower set, $\tau_L < \tau_L^{L_i/R_j}(\alpha)$. Then, by the construction of the magnitude-maximizing functions of α, we have $d_R\tau_R > d_A\tau_A + d_B\tau_B$ (i.e., LHS is decreasing in α, and RHS is increasing in α). To keep the difference in signs, we must have

$$d_R\tau_R > d_R\chi_R = d_A\chi_A + d_B\chi_B > d_A\tau_A + d_B\tau_B.$$

This implies $\chi_R < \tau_R$, $\chi_A > \tau_A$, and $\chi_B > \tau_B$. Now, consider an upper set, $\tau_L > \tau_L^{L_i/R_j}(\alpha)$. Then, $d_R\tau_R < d_A\tau_A + d_B\tau_B$. To keep the difference in signs, we must have

$$d_R\tau_R < d_R\chi_R = d_A\chi_A + d_B\chi_B < d_A\tau_A + d_B\tau_B.$$

As a result, $\chi_R > \tau_R$, $\chi_A < \tau_A$, and $\chi_B < \tau_B$. The third case of equality is obvious. $\square$

A.10 Proof of Proposition 8 (2+1, asymmetric district)

The proof is in the three steps: First, we show that for $\alpha = \frac{1}{2}$ and $\tau_L \geqslant \frac{4}{7}$, both Ties 2 and 3 are relevant, and neither Tie 1 or Tie 4 is relevant. Second, we check the players' best responses. Third, we check for the stability of the equilibrium against small perturbations.

The first step is to prove that only Ties 2 and 3 are relevant pivot ties:

- Tie 1 is constrained: Since $\tau_L > \frac{1}{2} \geqslant \tau_L^{L_1/R_0}$, we are in the upper set where by Lemma 1, the ballots for unconstrained magnitudes satisfy $\chi_A < \tau_A$ and $\chi_B < \tau_B$. For $\alpha = \frac{1}{2}$, we have $\chi_A = \chi_B^2 \frac{2}{\tau_L} = \chi_B \frac{\chi_B}{\tau_B} < \chi_B$. This violates the constraint $S_{R_1} = x_R - x_B \geqslant x_R - x_A = S_{R_0}$.

- Tie 2 is unconstrained: For $\alpha = \frac{1}{2}$, we have $\tau_L^{L_0/R_1} = \tau_L^{L_0/R_0} = \frac{4}{7}$. Thus, for $\tau_L > \frac{4}{7}$, we are in the upper set for both Ties 2 and 3, and $\chi_A < \tau_A$ and $\chi_B < \tau_B$. For $\alpha = \frac{1}{2}$, we have $\chi_B = \chi_A^2 \frac{2}{\tau_L} = \chi_A \frac{\chi_A}{\tau_A} < \chi_A$. This complies with the constraint $S_{R_1} = x_R - x_B \geqslant x_R - x_A = S_{R_0}$.

- Tie 3 is unconstrained: For $\alpha = \frac{1}{2}$, we have $\chi_A = \chi_B^2 \frac{2}{\tau_L} = \chi_B \frac{\chi_B}{\tau_B} < \chi_B$. This complies with the constraint $S_{R_0} = x_R - x_A \geqslant x_R - x_B = S_{R_1}$.

- Tie 4 is constrained: Since $\tau_L > \frac{1}{2} \geqslant \tau_L^{L_1/R_1}$, we are in the upper set where by Lemma 1, the ballots for unconstrained magnitudes satisfy $\chi_A < \tau_A$ and $\chi_B < \tau_B$. For $\alpha = \frac{1}{2}$, we have $\chi_B = \chi_A^2 \frac{2}{\tau_L} = \chi_A \frac{\chi_A}{\tau_A} < \chi_A$. This violates the constraint $S_{R_0} = x_R - x_A \geqslant x_R - x_B = S_{R_1}$.

- Tie 1 and Tie 4 have equal constrained magnitudes for $\alpha = \frac{1}{2}$. We know that the constrained magnitude of Tie 1 is maximized for $x := x_A = x_B$, and consequently $x_R = 3x$. For Tie 4, we again maximize for $x := x_A = x_B$, and consequently $x_R = 3x$. Given symmetry $\alpha = \frac{1}{2}$, the problem is symmetric, and $mag(piv_{L_1/R_0}) = mag(piv_{L_1/R_1}) = 5\chi - 1$, where $\chi = \frac{1}{2}\sqrt[5]{\frac{\tau_L \tau_L (1-\tau_L)^3}{27}}$.

- Tie 2 and Tie 3 have equal magnitude for $\alpha = \frac{1}{2}$. It is easy to see that if $\tau_A = \tau_B$, the magnitude-maximization problems are symmetric, hence $\chi_A^{L_0/R_1} = \chi_B^{L_0/R_0}$, $\chi_B^{L_0/R_1} = \chi_A^{L_0/R_0}$ and $\chi_R^{L_0/R_1} = \chi_R^{L_0/R_0}$, hence also $mag(piv_{L_0/R_1}) = mag(piv_{L_0/R_0})$.

- Comparing magnitudes: It remains to compare the constrained magnitude of Tie 1 with the unconstrained magnitude of Tie 2. Comparing magnitudes is extremely difficult, but we can exploit that even the constrained magnitude of Tie 2 is higher that the constrained magnitude of Tie 1. Namely, a constrained magnitude of Tie 2 is $\frac{7}{2}\hat{\chi}$, where $\hat{\chi} = \frac{1}{2}\sqrt[7]{\frac{8\tau_L^4(1-\tau_L)^3}{27}}$. The inequality $\frac{7}{2}\hat{\chi} > 5\chi$ boils down to $\left(\frac{1-\tau}{\tau}\right)^6 < 2^{15}3^6 7^{35}$. We evaluate it for $\tau_L = \frac{4}{7}$ to obtain a true inequality of $1 < 2^{27}7^{35}$. Using that LHS is decreasing in τ_L, the inequality holds also for any $\tau_L > \frac{4}{7}$.

The second step is to verify the voters' best responses. Recall that in our strategy profile, candidate L_1 is not serious, and only the three candidates L_0, R_0, R_1 are serious candidates. Pivotal events are ballot-specific since using only a positive vote differently affects a near-tie than using both a positive and negative vote. We denote pivots for a sincere ballot without prime and pivots for a strategic ballot with prime. We define pivots only for Ties 2 and 3. For R-voter and Tie 2,

$$piv_{R_1/L_0} := \{S_{L_0} - S_{R_1} \in \{0,1,2\} \ \&\ S_{R_1} \geqslant S_{R_0}\},$$
$$piv'_{R_1/L_0} := \{S_{L_0} - S_{R_1} \in \{0,1\} \ \&\ S_{R_1} \geqslant S_{R_0}\}.$$

For R-voter and Tie 3,

$$piv_{R_0/L_0} := \{S_{L_0} - S_{R_0} \in \{0,1,2\} \ \&\ S_{R_0} \geqslant S_{R_1}\},$$
$$piv'_{R_0/L_0} := \{S_{L_0} - S_{R_0} \in \{0,1\} \ \&\ S_{R_0} \geqslant S_{R_1}\}.$$

As a consequence, $\Pr(piv_{R_1/L_0}) > \Pr(piv'_{R_1/L_0})$ and $\Pr(piv_{R_0/L_0}) > \Pr(piv'_{R_0/L_0})$. For L-voter and Tie 2,

$$piv_{L_0/R_1} := \{S_{L_0} - S_{R_1} \in \{-2,-1,0\} \ \&\ S_{R_1} \geqslant S_{R_0}\},$$
$$piv'_{L_0/R_1} := \{S_{L_0} - S_{R_1} \in \{-1,0\} \ \&\ S_{R_1} \geqslant S_{R_0}\}.$$

For L-voter and Tie 3,

$$piv_{L_0/R_0} := \{S_{L_0} - S_{R_0} \in \{-1,0\} \ \&\ S_{R_0} \geqslant S_{R_1}\},$$
$$piv'_{L_0/R_0} := \{S_{L_0} - S_{R_0} \in \{-2,-1,0\} \ \&\ S_{R_0} \geqslant S_{R_1}\}.$$

As a consequence, $\Pr(piv_{L_0/R_1}) < \Pr(piv'_{L_0/R_1})$ and $\Pr(piv_{L_0/R_0}) > \Pr(piv'_{L_0/R_0})$.

For each $t = L,R$, we introduce the values of the gains for casting a particular ballot v^t in our profile, $G_t(v_t)$. Since we evaluate the gains for $n \to \infty$, we can omit the negligibly small probabilities of Ties 1 and 4:

$$G_R(0,-1,1,1) \approx V \Pr(piv_{R_0/L_0}) + (V+1)\Pr(piv_{R_1/L_0}),$$
$$G_R(-1,0,1,1) \approx V \Pr(piv'_{R_0/L_0}) + (V+1)\Pr(piv'_{R_1/L_0}),$$
$$G_L(1,1,0,-1) \approx V \Pr(piv_{L_0/R_0}) + (V-1)\Pr(piv_{L_0/R_1}),$$
$$G_L(1,1,-1,0) \approx V \Pr(piv'_{L_0/R_0}) + (V-1)\Pr(piv'_{L_0/R_1}).$$

- For R-voter, a sincere ballot clearly weakly dominates any other ballot since it supports the two most preferred serious candidates R_0, R_1 and punishes the least preferred serious candidate L_0. More formally, since $\Pr(piv_{R_1/L_0}) > \Pr(piv'_{R_1/L_0})$ and $\Pr(piv_{R_0/L_0}) > \Pr(piv'_{R_0/L_0})$, we have clearly $G_R(0,-1,1,1) - G_R(-1,0,1,1) > 0$.

- The indifference of L-voter is possible because strategic ballot is more valuable in Tie 2, but sincere ballot is more valuable in Tie 3, and both ties have an equal magnitude. To make L-voter exactly indifferent for any sufficiently large n, we have to have

$$\frac{V}{V-1} = \frac{\Pr(piv'_{L_0/R_1}) - \Pr(piv_{L_0/R_1})}{\Pr(piv_{L_0/R_0}) - \Pr(piv'_{L_0/R_0})}.$$

- It is valuable now to see that $\Pr(piv'_{L_0/R_1}) - \Pr(piv_{L_0/R_1}) = \Pr(S_{R_1} = S_{L_0} + 2 \geqslant S_{R_0})$ and $\Pr(piv_{L_0/R_0}) - \Pr(piv'_{L_0/R_0}) = \Pr(S_{R_0} = S_{L_0} + 2 \geqslant S_{R_1})$.

- We seek a sequence of profiles τ_n where each is characterized by an increment ε_n ($\lim_{n\to\infty} \varepsilon_n = 0$) whereby $\frac{\tau_L}{2} + \varepsilon_n$ and $\frac{\tau_L}{2} - \varepsilon_n$ are the expected vote shares of sincere and strategic L-votes for each n. Intuitively, for each finite Poisson game with the expected number of voters n, the gradually decreasing increment makes L-voter exactly indifferent between a strategic and sincere ballot. From the indifference, we clearly see that the expected score of R_1 must be slightly larger than the expected score of R_0, thus $\varepsilon_n > 0$.

- By offset theorem (Myerson, 2000), we have that for any event (χ_A, χ_B, χ_C), the sensitivity of the probability of the event under profile $(n(\frac{\tau_L}{2} + \varepsilon_n), n(\frac{\tau_L}{2} - \varepsilon_n), n\tau_R)$ is written as

$$\frac{\partial \log \Pr(\chi_A, \chi_B, \chi_C | n\tau_n)}{\partial \varepsilon_n} = \left(\frac{\chi_A}{\tau_A} - 1\right) n - \left(\frac{\chi_B}{\tau_B} - 1\right) n.$$

  Consider now the close race $S_{R_1} = S_{L_0} + 2 \geqslant S_{R_0}$, which L-voter can change from a tie to a win by a strategic vote. Following Myerson (2002), we exploit in the limit of our equilibria, all probability of this event becomes concentrated, where the strategic votes disappear, and the offset ratios are $\frac{\chi_A}{\tau_A} = 1$ and $\frac{\chi_B}{\tau_B} = 0$. Hence,

$$\frac{\partial \log \Pr(S_{R_1} = S_{L_0} + 2 \geqslant S_{R_0} | n\tau_n)}{\partial \varepsilon_n} = n.$$

  Similarly, consider the close race $S_{R_0} = S_{L_0} + 2 \geqslant S_{R_1}$, which L-voter can change from a tie to a win by a sincere vote. In the limit of our equilibria, all probability of this event becomes concentrated, where the sincere votes disappear, and the offset ratios are $\frac{\chi_A}{\tau_A} = 0$ and $\frac{\chi_B}{\tau_B} = 1$. We have:

$$\frac{\partial \log \Pr(S_{R_0} = S_{L_0} + 2 \geqslant S_{R_1} | n\tau_n)}{\partial \varepsilon_n} = -n.$$

- We now put the indifference equation into logarithm, and seek $\log \Pr(S_{R_1} = S_{L_0} + 2 \geqslant S_{R_0} | n\tau_n) - \log \Pr(S_{R_0} = S_{L_0} + 2 \geqslant S_{R_1} | n\tau_n = \log \frac{V}{V-1}$. Following Myerson (2002), this amounts to $(n - -n)\varepsilon_n = \log \frac{V}{V-1}$. As a result, $\varepsilon_n = \frac{1}{2n} \log \frac{V}{V-1}$.

Finally, we see that for $\tau_L > \frac{4}{7}$, the equilibrium is stable.

- By Lemma 2, if $\alpha = \frac{1}{2}$ and $\tau_L > \frac{4}{7}$, we are in the upper sets of both Ties 2 and 3.

- In the upper set of Tie 2, notice that the respective unconstrained magnitude *mag* (piv^*_{L0/R_1}) behaves in the following way: In an upper set, $(1+\alpha)\tau_L > \chi_A + 2\chi_B = 2\chi_C > 2(1-\tau_L)$. An increase in α increases the difference $(1+\alpha)\tau_L - \chi_A - 2\chi_B$ while leaving the difference $2\chi_C - 2(1-\tau_L)$ unchanged. This means that with an increase in α, the probability mass for the pivot decreases, and the pivot has a lower magnitude.

- Similarly, in the upper set of Tie 3, the respective unconstrained magnitude *mag* (piv^*_{L0/R_0}) behaves in the following way: In an upper set, $(2-\alpha)\tau_L > 2\chi_A + \chi_B = 2\chi_C > 2(1-\tau_L)$. An increase in α decreases the difference $(2-\alpha)\tau_L - \chi_A - 2\chi_B$ while leaving the difference $2\chi_C - 2(1-\tau_L)$ unchanged. This means that with an increase in α, the probability mass for the pivotal event increases, and the pivot must have a higher magnitude.

- Consider a small deviation $\varepsilon > 0$. For $\alpha = \frac{1}{2} - \varepsilon$, we have *mag* $(piv_{L0/R_1}) >$ *mag* (piv_{L0/R_0}). L-voters deviate by voting strategically (increasing α). For $\alpha = \frac{1}{2} + \varepsilon$, we have *mag* $(piv_{L0/R_0}) >$ *mag* (piv_{L0/R_1}). L-voters deviate by voting sincerely (decreasing α). □

A.11 Proof of Proposition 9 (Non-mixed equilibrium, asymmetry)

We begin with the symmetric profiles of 2+0 and AV, where $v_L = (1,1,0,0)$. Scores are $S_{L_1} = S_{L_0} = x_L$ and $S_{R_1} = S_{R_0} = x_R$, and a unique pivot tie is characterized by $x_L = x_R$. (By offset theorem, the relative probability of the lower near tie ($x_L = x_R - 1$) to tie is $\sqrt{\frac{\tau_R}{\tau_L}}$, and the relative probability of the upper near tie ($x_L = x_R + 1$) to tie is $\sqrt{\frac{\tau_L}{\tau_R}}$.) The following table yields L-voter's gains for each event:

Ballot		$x_L = x_R - 1$	$x_L = x_R$	$x_L = x_R + 1$
Triple vote	$(1,1,1,0)$	$\frac{4V-1}{6}$	$\frac{V-1}{3}$	0
Sincere vote	$(1,1,0,0)$	V	V	0
Quality vote	$(1,0,1,0)$	$\frac{V+1}{2}$	1	$\frac{1-V}{2}$

Since $V > \max\{\frac{4V-1}{6}, \frac{V+1}{2}, \frac{V-1}{3}, 1\}$, and $\frac{1-V}{2} < 0$, a sincere vote is clearly a unique best response. The profile is not an equilibrium.

We now turn to the symmetric profiles of CAV, where $v_L = (1,1,-1,-1)$. L-voter's gains for each event are identical, only the triple vote is $(1,1,1,-1)$, the sincere vote is $(1,1,-1,-1)$, and the quality vote is $(1,-1,1,-1)$. It is easy to see that any other ballot that does not use all negative votes is dominated by a ballot that uses all negative votes; hence, the sincere ballot $(1,1,-1,-1)$ is a unique best response. The profile is not an equilibrium either. □

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