A Short-Horizon Model of Asset Pricing: Equilibrium Analysis[1]

by

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Abstract

This paper analyzes a temporary financial market equilibrium by considering a two-period model of asset pricing with s securities, one riskless bond, and a continuum of heterogeneous agents with different preferences, endowments, and beliefs. Investors' objectives are to maximize the expected utility of the next period wealth. In this paper, after making certain assumptions, I show the existence of a competitive financial market equilibrium.

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1. Introduction

The concept of a competitive equilibrium - one of the fundamental concepts in economic theory - is based on the Walras' idea of "allocation of resources through the price system." This concept is based on the assumption that decision makers in the economy take a price system as given and make their decisions independently of each other. The only link between individuals' decisions is the price system. Traditional competitive equilibrium models consider finitely many traders in the market. In this case, the influence of a given participant on the market price is not really zero. If there are a very large number of participants in the market, then the influence of each individual on the market price becomes "negligible," i.e., each agent acts as a price-taker in the market to maximize his utility, subject to the budget constraint. In my model, I consider a continuum of agents, which means that a single agent has no effect on the total excess demand and thus on equilibrium prices. This is known as an atomless economy case. An alternative case to consider would be the economy with a continuum of agents and a number of "big players," who have infinitely more endowment than any agent in the atomless part. Aumann [5] was the first to consider an atomless economy with perfect competition. In his work the space of agents' characteristics is the Banach space. He proved the existence of a competitive equilibrium and showed the equivalence between core and set of competitive allocations.

Banach spaces are a natural choice for infinite dimensional problems. One can use the Riesz-Fischer representation theorem with the Parseval equation to show that the Banach space is analogous to the infinite dimensional Euclidean space.[2] To study a competitive equilibrium model with an infinite number of commodities, Connor [11], Bergstrom [8], Chichilnisky and Heal [12], for example, used weighted L_2 space as a commodity space. Bewley [9] was the first who studied the existence of a competitive equilibrium in an economy with a finite number of consumers and L_∞ commodity space. He proved the existence of an equilibrium if preferences are Mackey semi-continuous, and aggregate endowment is bounded. This result was generalized by Mas-Colell [31], who considered the case of finitely many consumers and described commodity space as a topological vector lattice.

In my model, I follow Aumann's idea and describe the space of investors' characteristics as the Banach space, which is the Cartesian product of three different spaces: the space of investors' utility functions, the space of investors' beliefs and the space of investors' endowments. My work is also related to that of Green [21],

[2] See Kolmogorov & Fomin [27], pp 149-154.

Grandmont [18], and Werner [36] in the sense that agents' beliefs, which is given by a probability measure, differ across agents. In my model, individual beliefs have the same support while this assumption is not necessary in the models of Green, Grandmont, and Werner. By assuming different supports behind individual beliefs, the authors show that the "overlapping expectations" condition is a sufficient condition for the existence of an equilibrium. In my model, assuming that agents have overlapping expectations, I can concentrate on individual beliefs as the major determinant of the equilibrium asset prices in the short run.

In his classical paper, Lucas [30] shows that the stream of future discounted real dividends determines the asset price. The key to his result is that he considers an infinitely lived representative agent who maximizes the sum of discounted utilities. In the model, equilibrium asset prices are functions of the current level of outputs produced in the economy. Assuming that agents know true joint probability distribution of the output generated process, they can correctly (rationally) forecast the joint distribution of the future prices. The rational expectations require that the price function implied by consumers' behavior be the same as the price function on which agents' decisions are based. In contrast the present paper emphasizes the case in which investors in the market are imperfectly informed about the factors (in this model, other investors' characteristics) which determine asset prices. Since investors can't perfectly forecast the distribution of future prices, they form subjective prior beliefs about it.[3] In the rational expectation case, agents' subjective priory beliefs are identical with the true objective distribution function which generates the asset price process. However, in this paper subjective distributions necessarily differ from the true distribution. Differences in subjective beliefs allow investors to participate in security trading for speculation and to find short - run gains from price changes. In reality short – run gains or losses from changes of nominal asset prices are more significant than gains or losses from real dividends. Therefore it is necessary to analyze important factors behind the short - run asset price movements. The aim of my work is to do this.

My model is closely related to the work of Hart [22], who considers a two - period competitive equilibrium model of asset pricing with s securities and one riskless bond in the market. His economy consists of a finite number of investors with different beliefs. Their objective is to maximize the expected utility of the next period wealth. Under the conditions that investors are strictly risk averse and investors' probability beliefs have a compact support, the author proves the

[3] See for example Berger [7] and Bawa, Brown, and Klein [6] for a general introduction to Bayesian analysis and its application to portfolio theory.

existence of a competitive financial market equilibrium. The major difference between his work and my model is that I consider, for the reasons given above, a continuum of investors in the market.

In general, the setting of the model is as follows. I consider a financial market with a continuum of investors. Traders have different preferences, different endowments, and different information about other agents' characteristics. The difference of information among investors causes them to form different expectations about future prices. Each investor's financial decisions are made under uncertainty about next period security prices. If an investor purchases shares of stocks, then some other investor sells them. Both investors make decisions in a rational manner. Perhaps, the buyer/seller thinks that there is a high probability that the future price of the shares will go up/high probability that the future price of the shares will go down. Thus, one distinguishes three main reasons for trading: different expectations, different risk aversion, and different endowments. The model considers the bond market and s securities markets. With initial endowments given and initial beliefs exogenously predetermined, asset prices in those markets and agents' income are simultaneously endogenously determined.[4] This model explains short-run behavior of asset prices when asset returns and consequently agents' income come only from price changes not from dividends. An investor's objective is to maximize the expected utility from the next period wealth, which is determined as the market value of his portfolio. Investors in the markets make decisions about optimal assets and bond holdings using their preferences, their information and initial portfolio holdings. In the model it is assumed that in the short run, when investors can find significant gains or losses from security trading, a consumption decision can have a negligible effect on an agent's utility and consequently on his behavior. This is why consumption decision is ignored in the study.

My paper is organized as follows. In the first section, I formulate the model and make assumptions about agents' utility, endowments, and probability beliefs. In the second section, I define the topological structure on the space of investors' characteristics. In section three, I prove the existence of competitive financial market equilibrium prices. Finally, I make concluding remarks and identify directions for future research.

[4]In the model it is not explicitly described who issues securities and who pays dividends on them. It is assumed that there are s different risky assets and one riskless bond in the market available for trade.

2. Economic Model

Consider an economy which lasts two periods. There are $(s+1)$ securities markets and a continuum of investors in the economy. In the first period each investor is endowed with an initial portfolio and has expectations about the second-period stock prices. Traders can invest their own wealth in s risky securities, which are traded in the stock markets, and in one riskless bond, which is traded in the bond market. The investments are made under uncertainty about the next period prices. As a result, individual financial decisions in the markets are based on probabilistic beliefs about future prices. Each investor is initially endowed with a fixed number of securities and has as his objective maximizing the expected utility of the next period wealth. The next period wealth is defined as the next period market value of portfolio holdings. Let an abstract space $(I, \aleph)$ be the investors' space, where I is an investors' set and $\aleph$ is a collection of all possible investors' coalitions included in an empty set. Let us define an atomless probability measure μ on the investors' space $(I, \aleph)$. Each investor in the space is described by three different characteristics.

Let $p^* = (p_1^*, p_2^*, ..., p_{s+1}^*) \in \nabla^s$ denote current security prices normalized on the s dimensional unit simplex,[5] and $p = (p_1, p_2, ..., p_{s+1}) \in \nabla^s$ denote security prices in the second period. Let $B(\nabla^s)$ denote the Borel σ algebra on ∇^s. The space $(\nabla^s, B(\nabla^s))$ is the event space and each element of this space represents possible price vector in the second period. A decision maker's uncertainty about the next period prices can be represented by a probability measure on the space: $(\nabla^s, B(\nabla^s))$. Let $M(\nabla^s)$ denote a set of all probability measures on the space $(\nabla^s, B(\nabla^s))$. Let $\hat{U}$ denote the set of all utility functions with R_+ as its domain, and R_+ as its range. The space of agents' initial endowments is R_+^{s+1}.

Suppose that in the beginning of the first period, investor i holds portfolio

$$\overline{x}_i = (\overline{x}_{i1}, \overline{x}_{i2}, ..., \overline{x}_{is+1})$$

for each $i \in I$. The second period wealth is defined as $W_i = \sum_{j=1}^{s+1} p_j x_{ij}$ for each $i \in I$, where W_i is i-th investor's second period wealth, j denotes the ordinal number

[5] It will be shown below that a demand correspondence for each investor is homogenous of degree zero with respect to the first period security prices p^*. This means that a proportional increase or decrease of all prices in the market does not alter relative prices and thus leaves the demand correspondence unchanged. So one can normalize each period security price vector to s dimensional unit simplex.

of a security, p_j is its price, and x_{ij} is the quantity of the security j held by the investor i as a result of trading in the first period.

The following assumptions on the utility function of each investor are imposed:

(A1) If $W_i > W_i^{'}$, then $u_i(W_i) > u_i(W_i^{'})$ for each $i \in I$;

(A2) $u_i(W_i)$ is a twice continuously differentiable function for each $i \in I$;

(A3) $u_i(W_i)$ is a concave function for each $i \in I$.

The following assumption on each investor's beliefs is imposed:

(A4) An investor's beliefs are represented by a continuous probability distribution function with the support $(\nabla^s, B(\nabla^s))$, $supp(F_i) = (\nabla^s, B(\nabla^s))$.

Investors' preferences over the second period income is represented by the von Neumann-Morgenstern expected utility function.

Next, assumptions will be made about the probability measure on the investors' space and on the aggregate initial holding of securities in the economy:

(A5) $\mu(.)$ is an atomless measure in the investors' space, which means that $d\mu\left(i\right) = 0$ for all $i \in I$;

(A6) $\int_I \overline{x}_{ij}\, d\mu(i)$ is finite for each security $j = 1, 2, ..., s+1$;

(A7) short sales are not allowed in the economy.

Assumption (A5) states that the measure of each investor is zero. Assumption (A6) says that mean holdings of each asset are bounded from above. (A7) is self-contained.

Next define the budget set of the agent i. Let $\beta_i\left(p^*, \overline{x}_i\right)$denote the set of all feasible next period portfolio holdings when market prices are denoted by the vector p^* defined above, and initial endowments are $\overline{x}_i$. The budget set is

$$\beta_i\left(p^*, \overline{x}_i\right) = \left\{x \in R_+^{s+1} \mid p^* x_i \leq p^*\ \overline{x}_i\right\}. \tag{2.1}$$

An investor's objective is to maximize the expected utility of tomorrow's wealth, subject to the budget constraint:

$$\begin{array}{c} \max \quad \int_{\nabla^s} u_i\left(p x_i\right) dF_i\left(p\right) \\ s.t.\ x_i \in \beta_i\left(p^*, \overline{x}_i\right), \end{array} \tag{2.2}$$

where $F_i\left(p\right)$ is investor i's belief about tomorrow's prices. An investor i's optimization problem yields the demand correspondence for securities:

$$\xi_i\left(p^*, \overline{x}_i, F_i\right) = \left\{ \begin{array}{c} x_i \in \beta_i\left(p^*, \overline{x}_i\right) | \forall y \in \beta_i\left(p^*, \overline{x}_i\right) \text{ we have} \\ \int_{\nabla^s} u_i\left(p x_i\right) dF_i\left(p\right) \geq \int_{\nabla^s} u_i\left(p y_i\right) dF_i\left(p\right) \end{array} \right\}, \tag{2.3}$$

where $\xi_i(p^*, \overline{x}_i, F_i)$ is the set of the best choices for investor i from his budget set. Expression (2.3) shows that the $(s+1)$-dimensional vector $x_i(p^*, \overline{x}_i, F_i)$ is the optimal portfolio demand for investor i, which means that there does not exist a portfolio $y_i(p^*, \overline{x}_i, F_i)$ belonging to the budget set which is better than x_i.

3. The Space of Agents' Characteristics

The aim of this section is to topologize the space of agents' characteristics (utility, endowments and beliefs) and to define a probability measure on it. This is needed, firstly, to establish an isomorphic relationship between the space of investors' characteristics and the investors' space, and secondly, to introduce a metric on the space of investors' characteristics which allows us to define distance between agents' characteristics and speak about closeness of preferences, beliefs and endowments.[6] The space of agents' characteristics can be considered as a Cartesian product of three spaces: the space of agents' utility functions, the space of agents' beliefs and the space of agents' endowments. Let us consider each space separately to introduce appropriate topological structures to make each of them complete and metrizable.

Recall that $\hat{U}$ denotes the space of all utility functions, which are increasing, continuous and concave, (A1), (A2) and (A3). $\hat{U}$ is an infinite dimensional space. To make $\hat{U}$ separable and locally compact it is natural to endow it with the Hausedorff [23] closed convergence topology. In this I follow Kanai [26].[7] According to Urysohn's Metrization theorem, necessary and sufficient conditions for a topological space to be metrizable are countability of basis and normality of the space.[8] Using the properties of the topology of closed convergence, it can be easily shown that $\hat{U}$ is the metrizable space.[9]

Next turn to the space of agents' endowments. Since there are $(s+1)$ securities traded in the market (s risky assets and one riskless bond), the space of agents' endowments must be a subspace of the positive orthrant of the $(s+1)$-dimensional Euclidean space: R_+^{s+1}. Let the topology on the space of agents' endowments be

[6] From an economic point of view, one can interpret "similarity" of investors' characteristics. For example, preferences are "similar" if investors have "similar" demands on securities with the same wealth, beliefs and in the same price situation.

[7] Kanai [26] studied continiuity properties of the core of an economy and convergence of agents' preferences and their demands with similar tastes.

[8] See Kolmogorov and Fomin [27], p.90 or Hildenbrand [24] for more details.

[9] For the proof, see for example Hildenbrand [24], pp. 96-97.

generated by Euclidean topology.[10] From this it follows that the space of agents' endowments has a finite basis.

Beside agents' utility function and endowment spaces there is one more space to consider - the space of agents' beliefs - $M(\nabla^s)$, which is the space of all probability measures of next period prices. Let the weak topology be introduced on this space. The idea of endowing the space of agents' beliefs with the weak topology is developed in Green [19, 21], Green and Stokey [20], and Allen [1] [2]. This is important to determine the weak convergence in the space of agents' beliefs.

Definition 3.1: A sequence of probability measures $F_{in}(p)$ converges weakly to the probability measure $F_i(p)$ if for any continuous utility function $u_i(px)$

$$\lim_{n\to\infty}\int_{\nabla^s} u_i(px)\,dF_{in}(p) = \int_{\nabla^s} u_i(px)\,dF_i(p)\,. \tag{3.1}$$

From definition 3.1 it follows that when agents have "similar" utilities and endowments, closeness of agents' beliefs about next period prices implies closeness of agents' expected utilities, and thus, their asset demands.

$M(\nabla^s)$ has a countable basis because the following theorem holds:

Theorem 1: "$M(X)$ is a separable metric space if and only if X is a separable metric space."[11]

In the case described in the model, ∇^s is the domain of all probability measures. ∇^s is a separable space. From this it follows that $M(\nabla^s)$ also has a countable basis.

Having defined the topologies of the subspaces of the space of agents' characteristics, one can define the product topology on the space of agents' characteristics: $\hat{U}\times R_+^{s+1}\times M(\nabla^s)$. Under the product topology the space of agents' characteristics is a complete, separable, metrizable space. One can use this important result from the measure theory.

Theorem 2: There exists always an atomless probability measure on an uncountable, complete, separable, metric space.[12]

So we can establish an atomless probability measure, ν, on the space of in-

[10] Debreu [16] introduced the Euclidian metric on the space of agents' endowments.

[11] Proof is given in Parthasarathy [32], p. 43, Theorem 6.2.

[12] For the proof see Parthasarathy [32], theorem 8.1. pp. 53-54.

vestors' characteristics.[13] The measurable space

$$\left(\hat{U} \times R_+^{s+1} \times M(\nabla^s), \Im\left(\hat{U}\right) \otimes \Im\left(R_+^{s+1}\right) \otimes \Im\left(M(\nabla^s)\right)\right)$$

represents the space of investors' characteristics.[14]

Having defined the space of investors' characteristics and a probability measure on it, one can use Skorokhod's theorem to show the existence of a continuous, one-to-one correspondence between the investors' space $(I, \aleph, \mu)$ and the space of investors' characteristics

$$\left(\hat{U} \times R_+^{s+1} \times M(\nabla^s), \Im\left(\hat{U}\right) \otimes \Im\left(R_+^{s+1}\right) \otimes \Im\left(M(\nabla^s)\right), \nu\right)$$

and additionally to consider a probability measure on the investors' space.

Theorem 3 (Skorokhod[15])**:**

Let T be a separable metric space and ν_n a weekly converging sequence of measures on T with limit ν. Then there exists a measure space $(\Omega, \Im, \lambda)$ and a measurable mapping f and f_n $(n = 1, 2, ...)$ of Ω into T, such that $\nu = \lambda \circ f^{-1}$, $\nu_n = \lambda \circ f_n^{-1}$ and $\lim\limits_{n \to \infty} f_n = f$ a.e. in Ω. Additionally if T is a complete separable metric space, then the measure space $(\Omega, \Im, \lambda)$ can be chosen to be the unit interval with the unit Lebesgue measure.

Skorokhod's theorem guaranties an isomorphism between investors' space and the space of their characteristics. In other words, a one-to-one correspondence between these spaces is defined: for each investor in the economy there corresponds only one point from the space of agents' characteristics, and vice versa.

In conclusion, it should be emphasized that this section makes an important step in modelling the economy by defining the probability measure on the space of investors' characteristics and establishing an isomorphic relationship between the investors' space and the space of investors' characteristics. In the next section of the paper, a formal definition of the economy will be given, and the existence of an equilibrium will be proved.

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[13] Since a probability measure was established on the space of agents characteristics' it becomes easier to make an economic interpretation. For example, if $A \subset \hat{U} \times R_+^{s+1} \times M(R_{++}^s)$, the number $\nu(A)$ is the fraction of investors who have characteristics belonging to the set A.

[14] The space $\left(\hat{U} \times R_+^{s+1} \times M(\nabla^s), \Im\left(\hat{U}\right) \otimes \Im\left(R_+^{s+1}\right) \otimes \Im\left(M(\nabla^s)\right)\right)$ denotes the product of measure spaces $\left(\hat{U}, \Im\left(\hat{U}\right)\right)$, $\left(R_+^{s+1}, \Im\left(R_+^{s+1}\right)\right)$, and $(M(\nabla^s), \Im(M(\nabla^s)))$, where sigma algebras $\Im\left(\hat{U}\right)$, $\Im\left(R_+^{s+1}\right)$, and $\Im(M(\nabla^s))$ are generated by the closed convergence, Eucledean and week convergence topologies respectively.

[15] For the proof of the theorem, see Skorokhod [34].

4. Existence of an Equilibrium

Two definitions of the economy and a competitive equilibrium in it are followed by the theorem showing that conditions A1-A7 guarantee the existence of the competitive equilibrium.

Definition 4.1. An exchange economy ε is a correspondence between the investors' space $(I, \aleph, \mu)$ and the space of investors' characteristics:

$$(I, \aleph, \mu) \xrightarrow{f} \left(\hat{U} \times R_+^{s+1} \times M\left(\nabla^s\right), \Im\left(\hat{U}\right) \otimes \Im\left(R_+^{s+1}\right) \otimes \Im\left(M(\nabla^s)\right), \nu\right). \quad (4.1)$$

For each set $A \subset I$ from the investors' space I, we have $f\left(A\right) = B \subset \hat{U} \times R_+^{s+1} \times M(R_{++}^s)$ and the image measure, μ, of the measure ν, with respect to the mapping f^{-1} is the relative fraction of investors (probability measure of the set A) who have characteristics belonging to $B \subset \hat{U} \times R_+^{s+1} \times M(\nabla^s)$. Thus, $\nu\left(B\right) = \mu\left(f^{-1}\left(B\right)\right)$. The marginal distribution of ν on $\hat{U}$, the marginal distribution of the measure ν on R_+^{s+1}, and the marginal distribution of ν on $M(\nabla^s)$ are called distribution of utility functions, distribution of investors' initial portfolios, and distribution of investors' beliefs respectively. In general, it is not assumed that these distributions are independent.

Let us define the mean demand and mean excess demand of securities. Recall that expression (2.3) defines the demand correspondence, $\xi_i : R_+^{s+1} \times M\left(\nabla^s\right) \times \nabla^s \to R_+^{s+1}$, for each investor i. Given a defined probability measure on the investors' space, $\mu(i)$, one can define mean excess demand correspondence on securities:

$$\phi\left(p^*\right) = \int_I \xi_i\left(p^*\right) d\mu\left(i\right) - \int_I \overline{x}_i \, d\mu\left(i\right). \quad (4.2)$$

Definition 1.3.2. A competitive equilibrium in the economy ε is a pair of a price vector $p^* \in \nabla^s$ and an allocation function $x_i^* : I \to R_+^{s+1}$, such that

1) for almost every investor i, x_i^* is the best choice, given an investor's utility, beliefs, and endowment and price vector p^*; $x_i^* \in \xi_i\left(p^*\right)$;

2) mean demand equals the mean supply for each security market,

$$\int_I x_i^* d\mu\left(i\right) = \int_I \overline{x}_i \, d\mu\left(i\right). \quad (4.3)$$

The demand correspondence of each investor is dependent on the price vector. It is clear that individual demand is homogeneous of degree zero with respect to the first period security prices: a $\lambda > 0$ times increase of all prices in the market does not alter the relative prices, and thus, leaves the demand correspondence unchanged. Since the demand correspondence is homogeneous of degree zero with respect to prices, it is possible to restrict the search for an equilibrium price vector to an interior of the s dimensional unit simplex.[16]

$$P = \left\{ p^* \in R_{++}^{s+1} \mid \sum_{j=1}^{s+1} p_j^* = 1 \right\} \tag{4.4}$$

To prove the existence of the competitive equilibrium, Debreu's theorem will be applied. To prepare the setting for the theorem application, consider the following properties of the excess demand correspondence.

1. The mean excess demand correspondence, $\phi(p^*)$, is convex-valued.

Proof:

From the expression (4.2), it follows that

$$\phi(p^*) = \int_I \xi_i(p^*)\, d\mu(i) - \int_I \overline{x}_i\, d\mu(i).$$

The assumptions (A1)-(A3) imply that an individual excess demand correspondence, $\xi_i(p^*)$, is a closed and convex - valued correspondence. (See Hildenbrand [24], chapter 1.2, Proposition 3.) Since individual demand correspondence is a closed and convex - valued correspondence, one can apply Aumann's [4] result to conclude that the mean demand correspondence $\int_I \xi_i(p^*)\, d\mu(i)$ and consequently the mean excess demand correspondence is also closed and convex - valued. Q.E.D.

2. $\phi(p^*)$ is bounded from below.

Proof:

Consider the optimal portfolio, $x_i \in \xi(p^*)$, for investor i from his optimal demand correspondence $\xi_i(p^*)$. The desirability assumption (A1) implies that an optimal portfolio always satisfies the budget constraint:

$$p^* x_i = p^*\, \overline{x}_i\ .$$

Let the portfolio norm for investor i's portfolio be denoted by

$$\|x_i\| = \sum_{j=1}^{s+1} |x_{ij}|\,.$$

[16]Fifth property of a mean excess demand correspondence shows that the norm of the mean excess demand correspondence at the boundaries of the unit simplex becomes infinity. This allows us to ruled out the possibility of zero prices.

Then the following holds:

$$p_j^* x_{ij} \leq \sum_{j=1}^{s+1} p_j^* x_{ij} = \sum_{j=1}^{s+1} p_j^* \ \overline{x}_{ij} \leq \left(\sum_{j=1}^{s+1} p_j^* \right) \left(\sum_{j=1}^{s+1} \overline{x}_{ij} \right) = \sum_{j=1}^{s+1} \overline{x}_{ij} = \|\overline{x}_i\| . \qquad (4.5)$$

The first inequality in expression (4.5) shows that the amount of wealth invested in the j-th security does not exceed the total wealth of investor i. The second inequality is Holder's inequality. The last equality in this expression holds by definition of the initial portfolio norm. Hence, it was shown that $x_{ij} \leq \frac{\|\overline{x}_i\|}{p_j^*}$. On the other hand, $x_{ij} \geq 0$. Thus, for each investor i, demand for security j is bounded: $0 \leq x_{ij} \leq \frac{\|\overline{x}_i\|}{p_j^*}$. We have to note here that assumption (A4), that investors' beliefs aren't degenerate at zero, rules out the possibility of a zero security price, $p_j^* = 0$,[17] Then the correspondence

$$i \rightarrow \left(\xi_{i1} \left(p^* \right), \xi_{i2} \left(p^* \right), ..., \xi_{is+1} \left(p^* \right) \right) \qquad (4.6)$$

is bounded from below by zero and from above by the integrable function

$$i \rightarrow \left(\frac{\|\overline{x}_i\|}{p_1^*}, \frac{\|\overline{x}_i\|}{p_2^*}, ... \frac{\|\overline{x}_i\|}{p_{s+1}^*} \right).$$

Given that an individual demand correspondence is bounded from both sides, the mean excess demand correspondence is also bounded from below by

$$- \int_I \overline{x}_i \, d\mu \left(i \right),$$

which is finite by assumption (A6). Q.E.D.

3. $\phi \left(p^* \right)$ is upperhemicontinuous with respect to prices.

Proof:

For each fixed i, the correspondence $p^* \rightarrow \xi_i \left(p^* \right)$ is upperhemicontinuous.[18]

Let G denote a closed neighborhood of an interior point p of the unit simplex. Since p is an interior point, we can choose G set this way so that it totally belongs to the unit simplex. Let $\pi \left(G \right)$ denote the minimum security price in G:[19]

[17] Below we also show that if the price for one particular security j goes to zero, then demand for this security goes to infinity.

[18] See Handbook of Mathematical Economics[3], volume 2, p. 728.

[19] Since G is closed and finite dimensional, such a minimum exists.

$$\pi(G) = \min_{\substack{p^* \in G \\ j=1,...s+1}} p_j^* . \tag{4.7}$$

The assumption $G \subset P$ implies that $\pi(G)$ is strictly greater than zero. Hence, one can impose an upper bound on investor i's demand on securities:

$$x_{ij}(p^*) \leq \sum_{j=1}^{s+1} x_{ij}(p^*) \leq (s+1) \max_j |x_{ij}(p^*)| \leq (s+1) \max_j \frac{\|\overline{x}_i\|}{p_j^*} = (s+1) \frac{\|\overline{x}_i\|}{\pi(G)}. \tag{4.8}$$

The first and the second inequalities in the expression (4.8) are obvious, the third inequality follows from the above established fact that $x_{ij}(p^*) \leq \frac{\|\overline{x}_i\|}{p_j^*}$.

Knowing that the individual demands for each security is bounded $x_{ij}(p^*) \leq (s+1) \frac{\|\overline{x}_i\|}{\pi(G)}$ and the bound function is integrable, one can use Auman's theorem (1965) to show that the mean excess demand correspondence

$$p \to \int_I \xi_i(p^*) \, d\mu(i) - \int_I \overline{x}_i \, d\mu(i)$$

is upperhemicontinuous.

Q.E.D.

4. $\phi(p^*)$ **satisfies Walras' Law.**

Proof:

The budget constraint of each investor holds with equality, i.e., $p^*\xi_i(p^*, \overline{x}_i, F_i) = p^* \, \overline{x}_i$. By integrating this expression over the investors' space on both sides and rearranging terms, one gets:

$$p^* \int_I \xi(p^*) \, d\mu(i) - p^* \int_I \overline{x}_i \, d\mu(i) = p^*\phi(p^*) = 0. \tag{4.9}$$

Q.E.D.

5. $\phi(p^*)$ **satisfies a boundary condition: if at least one security price approaches zero, then the norm of demand vector for securities goes to infinity.**

$$p_n^* \overset{n\to\infty}{\rightarrow} p_0^* \in \partial P \Rightarrow \|\phi(p_n^*)\| \overset{n\to\infty}{\rightarrow} \infty$$[20]

Proof:

For an given investor i, his demand $x_i(p_n^*) \in \xi_i(p_n^*)$ satisfies the following inequality:

$$\|x_i(p_n^*)\| = \sum_{j=1}^{s+1} |x_{ij}(p_n^*)| \leq (s+1) \max_j \frac{\|x_i\|}{p_{jn}^*} \leq (s+1)\|x_i\| \left(\sum_{j=1}^{s+1} \frac{1}{p_{jn}^*} \right). \quad (4.10)$$

From this inequality it follows that the correspondence $i \rightarrow \|x_i(p_n^*)\|$ is bounded by an integrable function $(s+1)\|x_i\| \left(\sum_{j=1}^{s+1} \frac{1}{p_{jn}^*} \right)$. Now we can use Fatou's lemma:

$$\int_I \liminf_{n\to\infty\, k\geq n} \|x_i(p_k^*)\| \, d\mu(i) \leq \lim_{n\to\infty} \inf_{k\geq n} \int_I \|x_i(p_k^*)\| \, d\mu(i). \quad (4.1)$$

Using the property of an individual's demand function, we have

$$\lim_{n\to\infty} \|x_i(p_n^*)\| = \infty,^{21} \quad (4.12)$$

and consequently

$$\liminf_{n\to\infty\, k\geq n} \|\xi_i(p_k^*)\| = \infty. \quad (4.13)$$

(for more details, see Hildenbrand [24], pp. 103-104)

Then using (4.10), (4.11), (4.12) and (4.13), for any $y(p_n^*) \in \phi(p^*)$ the following is true:

$$\lim_{n\to\infty} \|y(p_n^*)\| = \lim_{n\to\infty} \left\| \int_I (x_i(p_n^*) - \overline{x}_i) \, d\mu(i) \right\| = \lim_{n\to\infty} \int_I \|x_i(p_n^*) - \overline{x}_i\| \, d\mu(i) \geq$$

$$\lim_{n\to\infty} \inf_{k\geq n} \int_I \|x_i(p_k^*)\| \, d\mu(i) - \int_I \|\overline{x}_i\| \, d\mu(i) \geq \int_I \liminf_{n\to\infty\, k\geq n} \|\xi_i(p_k^*)\| \, d\mu(i) - \int_I \|\overline{x}_i\| \, d\mu(i).$$

[20] Here p_n^* denotes a sequence of price vectors converging to the price vector p_0^*, which belongs to the boundary of the unit simplex.

[21] For more details, see lemma 4 in Arrow and Intriligator [3], pp.721-722.

Since

$$\int_I \liminf_{n\to\infty\, k\geq n} \left\|\xi_i\left(p_k^*\right)\right\| d\mu\left(i\right)$$

in the above inequality is unbounded and $\int_I \left\|\overline{x}_i\right\| d\mu\left(i\right)$ is finite, it follows that

$$\lim_{n\to\infty} \left\|y\left(p_n^*\right)\right\| = \infty.$$

Q.E.D.

Given that properties (1)-(5) of the excess demand correspondence $\phi\left(p^*\right)$ hold, Debreu's fixed point theorem can be applied to prove the existence of a competitive equilibrium in the economy.

Theorem 5 (Debreu[22]**):** If $\phi\left(p^*\right)$ is convex-valued, bounded from below, upperhemicontinuous, and if $\phi\left(p^*\right)$satisfies Walras' Law and the boundary condition, then there exists a price vector $p^* \in P$, such that $0 \in \phi\left(p^*\right)$.

5. Concluding Remarks and Directions for the Future Research

This paper analyses the properties of an excess demand correspondence in financial markets of s risky securities, one riskless bond, and a continuum of heterogenous investors, and proves the existence of a temporary partial financial market equilibrium in the economy.

The notion of the mean excess demand correspondence in an economy requires a probability measure to be established on the space of investors' characteristics, which is a Cartesian product of the spaces of investors' utilities, beliefs, and endowments. Theorem 2 guarantees the existence of an atomless probability measure on the space of investors' characteristics because a topological structure was specified on this space - closed convergence topology on the space of investors' utility functions, weak convergence topology on the space of investors' beliefs and Euclidean topology on the space of investors' endowments - such that the space of investors characteristics becomes a complete, separable, metric space. Then using Skorokhod theorem, it was shown that, firstly, the investors' space with the unit Lebesgue measure exists, and secondly, there is an isomorphic relationship

[22] Theorem and its proof are given in Border [10], pp 81-82, theorem 18.1.

between the investor's space and the space of investors' characteristics. This isomorphic relationship makes the measures defined on the investors' space and the space of investors' characteristics be uniquely determined from each other. From the economic point of view, this means that for any set of characteristics belonging to the space of the investors characteristics there is a uniquely determined set of investors who is described by these characteristics. Integrating over the investors' abstract space with respect to the measure established on it defines the mean excess demand correspondence in the economy (see equation (4.2)).

Then the paper studies the properties of the mean excess demand correspondence to show that they satisfy requirements of Theorem 5, and consequently, are sufficient for the existence of the temporary financial market equilibrium in the economy. In particular, given that 1) investors' utility functions are concave, monotone, and continuously differentiable, 2) the domain of the distribution of investors' beliefs is an s-dimensional unit simplex; 3) the mean endowment of each security in the economy is bounded; 4) the distribution function on the space of agents' characteristics is atomless; and 5) short-selling of securities is not allowed, the excess demand correspondence for each security was shown to be convex-valued, bounded from below, upperhemicontinuous, and to satisfy Walras' law and the boundary condition. Then using Theorem 5, it was proved that the equilibrium exists.

The main result of the paper is that it developed the theoretical basis for an economy with a continuum of heterogenous investors and opened the way for studying the sensitivity of asset prices with respect to changes in the investors' characteristics, in particular, investors' beliefs. A precise measure of the effect of investors' expectations about financial market fluctuations on asset prices will make it possible to quantify peculiarities of asset pricing in the short-run. Then introducing the real side of the economy in the model will enable one to determine temporary deviations of asset prices from their fundamental values. This will improve our understanding of "the stock market overreactions," and thus, bring us closer to the complete description of the way financial markets function.

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