Equilibrium Selection in Participation Games: A Unified Framework

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Abstract

In many applied settings, an activity or project requires a critical mass of participants to be worthwhile. This property can give rise to multiple equilibria. We study seven well-known equilibrium selection theories: two heuristic arguments, two models with rational players, and three from the evolutionary literature. With one exception, each relies on strategic complementarities. We weaken this to a mild single crossing property and show that the theories' predictions have a common form: an agent plays a best response to some fictional distribution of the participation rate of her opponents.

JEL: C72, C73, C79.

Keywords: equilibrium selection, evolutionary games, replicator dynamic, mutations, stochastically stable set, ergodic distribution, global games, dynamic games, unique implementation, Pareto criterion, single crossing properties, mechanism design.

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1 Introduction

Many activities offer benefits that depend on the number of others who participate. Examples include investing in a project; joining an online platform; leaving one's deposits in a bank; and buying an electric vehicle.

If participating is worthwhile *only* if enough others participate, then both all-participate and none-participate are self-fulfilling prophecies. This multiplicity poses a challenge for game theory. Economists have responded with a number of theories of how agents might select an equilibrium. These include heuristics such as the Pareto criterion, as well as the predictions of evolutionary and global games.

In applied work, most researchers have assumed that the agents use a *particular* selection criterion that the researcher chooses. A concern is that the resulting predictions may depend nontrivially on this choice. A more robust approach would be to prove results that hold for any selection criterion in some large set. However, a researcher who undertook this approach would need not only to grasp a variety of notational systems, but also to allocate scarce space in her paper to explaining and analyzing the various criteria. Unsurprisingly, most researchers have not done so.

In addition, many selection results rely on restrictive payoff assumptions that hamper their application. The most common is strategic complementarities: the payoff from participating is nondecreasing in the overall participation rate. This can easily be violated when a principal devises a scheme to induce the agents to participate.[1]

In summary, in order for a multi-criterion approach to be practical, two advances are needed. First, one must identify a class of criteria that have a common, parsimonious form that can be easily embedded in a larger model. Second, one must replace strategic complementarities with a weaker property that is easier to satisfy. This paper provides both such advances.

[1] Examples appear in Frankel [26, 27].

We study seven iconic selection theories from the literature and show that they all yield criteria of the same basic form: an agent plays a best response to some distribution of the proportion of others who participate. This distribution is common across the agents and does not depend on the game's payoffs. We refer to it as the agents' *fictional beliefs* as it need not coincide with the true distribution or with the agents' actual beliefs.

Furthermore, we assume only a weak single crossing property that Athey [5] calls "weak SC1": if the payoff from participating is positive for one other-agent participation rate, then it is not negative for any higher rate. This strengthens most of the theories we study, which have relied on the stronger property of strategic complementarities.

In particular, for the evolutionary theories of Foster and Young [23] and Kandori, Mailath and Rob [48], as well as the dynamic rational-player theory of Matsui and Matsuyama [53], the $n \geq 2$ player case was previously studied by Kim [50] under the assumption of strategic complementarities. We replace this with weak SC1. For the evolutionary theory of Fudenberg and Harris [32], we extend their two-player result to $n \geq 2$ players relying only on weak SC1.

Our results also contribute to the theory of global games. Goldstein and Pauzner [34] (GP) show that there is a unique Nash equilibrium if signal errors are uniformly distributed and payoffs satisfy a strict single crossing property that Athey [5] calls "strict SC1". Morris and Shin [55, p. 70] (MS) find a unique *threshold* equilibrium for *general* signal errors under strict SC1. Szkup [66] finds a unique threshold equilibrium for uniform signal errors, for a specific payoff function that satisfies only *weak* SC1. Finally, we find a unique threshold equilibrium for *general* signal errors and a *general* payoff function under weak SC1.

In section 8, we provide an algorithm - the Heuristic Search Procedure (HSP) - for finding an optimal scheme to induce a group of agents to participate in a joint activity. Frankel [27] applies HSP to three different variants of a security design model. In two of these variants, the payoff functions violate strategic complementarities yet satisfy weak SC1. Frankel's main results, including the optimality of debt, are proved for general fictional beliefs. They thus hold not only for our six selection criteria, but also for any other criteria of the given form - including those not yet identified.

An assumption of HSP is that the agents' fictional beliefs do not vary with the scheme offered. Intuitively, the literature typically derives a given selection criterion from an explicit context in which the participation game is played. In practice, a principal would know this context and hence be able to predict the agents' behavior. The argument is thus no different from a situation in which a single criterion is derived from an explicit model: the principal knows the model and thus which criterion the agents will use.

2 The Model

The game is played by a set I of *ex-ante* identical agents. The agents may be either discrete ($I = \{1, ..., n\}$) or infinitesimal ($I = [0, 1]$). They have aggregate measure one, so the measure of a single agent is $1/n$ in the discrete case.[2]

The decision of each agent is whether or not to participate. From the perspective of a given agent, we define $\ell \in [0, 1]$ to be the *other-agent participation rate*: the proportion of the agent's opponents who participate. An agent's payoff is given by a utility function $u(i, \ell)$ where the agent's action i is either 1 (participate) or 0 (not participate). We define

$$\pi(\ell) = u(1, \ell) - u(0, \ell) \tag{1}$$

to be the net payoff from participating given the other-agent participation rate ℓ. We

[2]We must remain vague for now about the context in which the game of I agents is played, as well as the agents' information sets and degrees of rationality. For instance, in evolutionary games, the game is repeatedly played by random sets I of symmetrically informed but boundedly rational players who are drawn from a larger population. In global games, the game is played once with a fixed set I of imperfectly informed but fully rational players.

will refer to π as the *payoff function.*

Let R be the set of feasible other-agent participation rates; it equals $[0,1]$ in the infinitesimal case and $\lambda = \left\{\frac{i}{n-1} : i = 0, ..., n-1\right\}$ in the discrete case. Let $\Pi = \{\pi : R \to \Re\}$ be the set of all payoff functions π on R. A selection criterion (or "criterion" for brevity) is a rule that partitions Π into three sets. In one set, the agents participate; in another, they do not; and in a third (which typically has measure zero), the criterion makes no prediction:

Definition 1. A *selection criterion* is a real-valued function Ξ on Π. Agents with payoff function π who follow the criterion Ξ do (not) participate when $\Xi(\pi)$ is positive (resp., negative); they may do anything if it is zero.

We define a *selection theory* to be a rationale for a selection criterion. We study seven such theories. Two are heuristic arguments, two are based on rational players, and three come from the evolutionary literature. We will show that under each such theory, an agent chooses a best response to the belief that the other-agent participation rate ℓ has some fixed distribution. That is, each theory yields a selection criterion of the form

$$\text{the agents (do not) participate if } \Xi(\pi) = \int_{\ell=0}^{1} \pi(\ell)\, d\Gamma(\ell) > (<)\, 0 \tag{2}$$

for some distribution Γ that depends on the theory. We refer to Γ as the agents' *fictional beliefs.*

In general, the distribution Γ does not coincide with the true distribution of other-agent participation rates.[3] Neither does it reflect the agents' beliefs over this distribution. Rather, it merely measures the relative importance of different segments of the payoff function π in determining which equilibrium the agents will select.

[3]For generic payoffs π, the integral in (2) will be either positive or negative: either all agents will participate, or none will. The true distribution thus puts all of its weight on one or zero. While there are fictional beliefs with this property, most put positive weight on intermediate other-agent participation rates.

3 Single Crossing Properties

If agents are discrete, it will suffice that the payoff function π satisfy a weak single crossing property: if it is positive for some ℓ, then it is not negative for any higher ℓ. Letting S be any subset of $\Re$, the definition is as follows.

Definition 2. (Athey [5]) A function $h : S \to \Re$ satisfies *weak single crossing in a single variable* ("weak SC1") on S if for all $s_H > s_L$ both in S, $h(s_L) > 0$ implies $h(s_H) \geq 0$.

For a continuum of agents, we will typically require a stronger condition:

Definition 3. (Athey [5]) A function $h : S \to \Re$ satisfies *single crossing in a single variable* ("SC1") on S if for all $s_H > s_L$ both in S, $h(s_L) \geq 0$ implies $h(s_H) \geq 0$ and $h(s_L) > 0$ implies $h(s_H) > 0$.

Some results from the literature rely on an even stronger property:

Definition 4. (Athey [5]) A function $h : S \to \Re$ satisfies *strict single crossing in a single variable* ("strict SC1") on S if for all $s_H > s_L$ both in S, $h(s_L) \geq 0$ implies $h(s_H) > 0$.

These properties are illustrated in Figure 1. In the left panel, weak SC1 holds: negative values do not follow positive ones. However, SC1 fails since (a) negative values follow zeroes and (b) zeroes follow positive values. In the middle panel, SC1 holds but strict SC1 fails as zeroes follow zeroes. (Under strict SC1, only positive values can follow a zero.) In the third panel, all three properties hold.[4]

By and large, our results assume weak SC1 when agents are discrete and SC1 when they are infinitesimal. In contrast, the prior literature relies largely on strategic

[4]As the names suggest, strict SC1 implies SC1 which, in turn, implies weak SC1.

complementarities, which are stronger than any of the three versions of SC1.[5,6] Hence, our results constitute a substantial increase in flexibility.

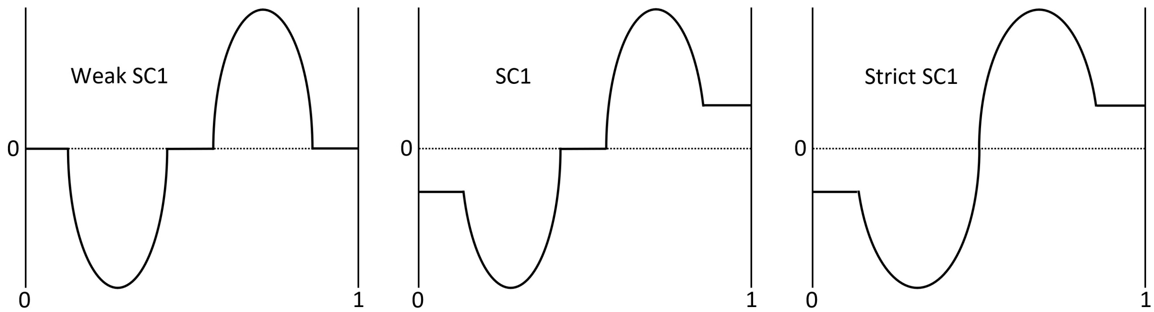

Figure 1: Sample functions $h : [0, 1] \to \Re$ that satisfy the three types of single crossing properties defined by Athey [5] and used in this paper.

We next prove a key technical result. Assume an agent plays the participation game against $n-1$ opponents, each of whom has an independent probability $z \in [0, 1]$ of participating. The probability that the other-agent participate rate faced by the given agent is $\ell \in \lambda$ is then the chance[7]

$$\kappa(\ell; z) = b\left(\lfloor \ell (n-1) \rfloor; n-1, z\right) \tag{3}$$

of $\lfloor \ell (n-1) \rfloor$ successes in $n-1$ independent trials, each with success probability z, where

$$b(i; n-1, z) = \begin{pmatrix} n-1 \\ i \end{pmatrix} z^i (1-z)^{n-1-i} \tag{4}$$

is the binomial density. The agent's expected payoff $\gamma(z)$ from participating is the

[5]More precisely, most prior selection results have assumed *strict* strategic complementarities (i.e., that the payoff function $\pi(\cdot)$ is increasing). This property implies strict SC1. In the global games literature, it is more common to assume only strategic complementarities, which means "no *decreasing* segments" (Frankel, Morris, and Pauzner [31, p. 4]). Strategic complementarities implies SC1, but is orthogonal to strict SC1 as the latter allows decreasing segments but not segments in which the function is identically zero.

[6]The only exception is global games; see section 1.

[7]The floor function in (3) ensures that the integral in (5) below is well defined. Nonintegral values of $\ell(n-1)$ receive no weight in that integral since they cannot occur in the discrete case; this is accomplished by setting the measure $\mu(\ell)$ to zero in (6).

sum, over $\ell \in \lambda$, of her realized payoff $\pi(\ell)$ weighted by this probability $\kappa(\ell; z)$, which we can write in three equivalent ways:[8]

$$\begin{aligned}\gamma(z) &= \sum_{\ell \in \lambda} \kappa(\ell; z)\, \pi(\ell) \\ &= \int_{\ell=0}^{1} \kappa(\ell; z)\, \pi(\ell)\, d\mu(\ell) \\ &= \sum_{i=0}^{n-1} b(i; n-1, z)\, \pi\left(\frac{i}{n-1}\right)\end{aligned} \tag{5}$$

where the measure μ on $[0,1]$ is

$$d\mu(\ell) = \begin{cases} 1 & \text{if } \ell \in \lambda; \\ 0 & \text{otherwise.} \end{cases} \tag{6}$$

That is, μ assigns unit (resp., zero) weight to ℓ if is feasible (resp., infeasible) given that the agent plays the game with exactly $n-1$ others. The following claim extends a result in Athey [74]:[9]

Claim 5. *If π satisfies weak SC1 on λ, then:*

1. *γ satisfies SC1 on $(0,1)$ and weak SC1 on $[0,1]$.*

2. *γ is Lipschitz-continuous on $[0,1]$ and agrees with π at the endpoints: $\gamma(0) = \pi(0)$ and $\gamma(1) = \pi(1)$.*

Claim 5 states that if each opponent has an independent probability z of participating and the realized payoff function π satisfies weak SC1 in ℓ, then the *expected* payoff function γ is Lipschitz-continuous and satisfies SC1 in z. These properties will play a key role in our treatment of the five noncooperative theories. To show that γ is an agent's expected payoff from participating, we rely on random signals in the

[8] The integral in (5) is well defined as equation (3) defines $\kappa(\ell; z)$ for all $\ell \in [0,1]$. But as $d\mu(s)$ is nonzero only if s is in λ, $\gamma(z)$ depends only on the values taken by $\kappa(\cdot, z)$ at integral values of $\ell(n-1)$.

[9] Proofs of all claims are in our online appendix (Frankel [28]).

global games theory and on random matching in the other four theories.[10]

4 Heuristic Theories

We begin with two heuristic theories. The first is the most optimistic of the seven theories:

The Pareto theory is the argument that the agents will choose "all participate" if it is a strict Nash equilibrium.[11]

We define an associated selection criterion:[12]

The Pareto criterion is the criterion given by (2) for the fictional beliefs[13]

$$\Gamma^{*}(\ell) \stackrel{d}{=} \begin{cases} 0 & \text{if } \ell < 1 \\ 1 & \text{if } \ell = 1. \end{cases} \tag{7}$$

That is, agents who rely on the Pareto criterion will play a best response to the belief that all others will participate.

The second heuristic theory is the most *pessimistic* of the various theories:

[10]For this reason, we will usually need π itself to satisfy SC1 and continuity when agents are infinitesimal: as play is against the entire population, there is no random matching process that would let us substitute the "nicer" payoff function γ.

[11]The Pareto theory is used to predict investor behavior in Chakraborty, Gervais, and Yilmaz [15]. The usual argument is that an agent can persuade others to choose the "good" equilibrium by stating her own intention to do so. Aumann [6] argues, to the contrary, that agents may have an incentive to lie. However, subsequent experimental evidence has suggested that people prefer truth-telling (Gneezy [33]) and that preplay communication makes the Pareto dominant outcome more likely (Blume and Ortmann [11]; Feri, Irlenbusch, and Sutter [22]).

[12]In addition to having an obvious heuristic basis, the Pareto criterion emerges in a variety of noncooperative contexts (Demichelis and Weibull [19]; Kim and Sobel [51]; Rabin [61]). However, we think it makes more sense to ground it in a heuristic than in a particular model. First, it is unclear which model to choose, unlike in the case of the Laplace, KMR, FY, and FH criteria which emerge in iconic settings. Second, the Pareto criterion has a long history in economics and an intuitive appeal that does not hinge on any particular model and that, moreover, seems more often to motivate models than to be motivated by them.

[13]The notation "$\stackrel{d}{=}$" indicates a definition.

The Unique-Implementation (UI) theory is a heuristic argument that states that the agents will choose "none participate" if it is a strict Nash equilibrium.[14]

We define an associated selection criterion:

The UI criterion is the criterion given by (2) for the fictional beliefs

$$\Gamma^{\mathrm{UI}}(\ell) = 1 \text{ for all } \ell. \tag{8}$$

That is, agents who rely on the UI criterion will play a best response to the belief that *no* others will participate.

The five noncooperative theories predict that the agents will coordinate. In contrast, the two heuristic theories do not rule out population mixing. To ensure consistency, we thus strengthen these heuristics by assuming that the agents, as a group, display a lexicographic preference for coordination:[15]

A1 If all-participate or none-participate is a Nash equilibrium, the agents will not mix.

With this addition, each heuristic theory is equivalent to the associated criterion under weak SC1:

Claim 6. *Assume A1. If π satisfies weak SC1, the Pareto theory implies the Pareto criterion and vice-versa.*

[14] The UI theory has been used in participation games by Bernstein and Winter [9], Halac, Kremer, and Winter [39], Halac, Lipnowski, and Rappoport [40], Segal [65], and Winter [70]. The usual argument is not that the agents will choose the bad equilibrium, but rather that the principal wishes to rule this out. The same approach is taken in dominant strategy implementation: the principal wishes to makes it dominant for the agents to truthfully reveal their types (Groves [37]; Green and Laffont [36]).

[15] E.g., for any $c \in (0,1)$, the payoff function $\pi(\ell) = \begin{cases} 0 & \text{if } \ell \leq c \\ -1 & \text{if } \ell > c \end{cases}$ satisfies weak SC1 and any participation rate in $[0, c]$ is a Nash equilibrium. The five noncooperative theories put positive weight on other-agent participation rates over c and thus predict that no agents will participate. It is less clear what the two heuristic theories would predict as the arguments in their favor assume two pure Nash equilibria. To pin down their predictions and ensure consistency with the noncooperative theories, we thus add assumption A1.

Claim 7. *Assume A1. If π satisfies weak SC1, the UI theory implies the UI criterion and vice-versa.*

5 Dynamic Theories

We next study four dynamic theories. In the discrete case, the participation game will be played by groups of $n \geq 2$ agents who are drawn at random from a much larger population.[16] If the proportion who participate in this larger population is $z \in [0, 1]$, then the probability that exactly $\ell \in \lambda$ randomly drawn opponents will participate is given by the function $\kappa(\ell; z)$ defined in (3). Our discussion of γ in section 3 then implies:

Claim 8. *In games in which groups of n agents are randomly matched from a large population to play the participation game, a player's expected static payoff from participating is given by $\gamma(z)$ defined in (5) where $z \in [0, 1]$ is the participation rate in the population.*

5.1 The Kandori-Mailath-Rob (KMR) Theory

The KMR theory is a discrete-time model in which random groups of n boundedly rational agents are selected, in each period, to play the participation game.[17] The action with the higher static payoff in a given period is chosen by more agents in the next. There are trembles: each agent has a small chance of choosing the suboptimal action.

More precisely, we study the following generalization of the evolutionary model of

[16] This population is finite in KMR and a continuum in the other three theories.

[17] The KMR theory is defined only for discrete agents.

Kandori, Mailath, and Rob [48].[18,19] The population consists of a large finite number N of agents. In each period $t = 0, 1, ...$, random groups of $n \geq 2$ agents are selected to play the participation game. By Claim 8, an agent's expected payoff from participating is thus $\gamma(z)$ where $z \in [0, 1]$ is the participation rate in the population. The model we study is in the following class.

Definition 9. (Ellison [21]; Young [72]) A Model of Evolution with Noise (MEN) is a triple $(Z, P, P(\varepsilon))$ consisting of a state space Z (a finite set); a Markov transition matrix P on Z that gives the transition probabilities in the absence of mutations; and a family of perturbed Markov transition matrices $P(\varepsilon)$ for each tremble probability $\varepsilon \in [0, \bar{\varepsilon}]$ such that (a) $P(\varepsilon)$ is ergodic[20] for all ε and (b) $P(\varepsilon)$ is continuous in ε with $P(0) = P$.

In KMR [48] without trembles, the agents play a static best response to the prior period's population participation rate z. This is a special case of the following class of deterministic dynamics, which we use instead.

Definition 10. (KMR [48]) The deterministic dynamic P in Def. 9 is *Darwinian* with respect to the relative payoff function γ if the following conditions hold.

1. If one action has a higher payoff under γ given the current participation rate z, the proportion playing that action rises in the next period (unless everyone is already playing that action, in which case they keep doing so).

2. If the two actions give the same payoff under γ given the current participation rate z, then the proportion playing a given action either (i) stays the same in the next period or (ii) could rise or fall, but does not move solely in one direction.

[18] KMR [48] prove their result for two-player, two-action coordination games. Kim [50] generalizes this result to coordination games with $n \geq 2$ players. We replace Kim's coordination game assumption with weak SC1 and relax KMR's [48] assumption on dynamics.

[19] While KMR [48] is often grouped with Young [72], the models are not equivalent (Jacobsen, Jensen, and Sloth [46]).

[20] A Markov chain is ergodic if every state is eventually reached from every other state with positive probability.

As in KMR [48], we assume further that the perturbed matrix $P(\varepsilon)$ is generated by independent random trembles (IRT): each agent plays according to the deterministic dynamic with probability $1-\varepsilon$ and randomizes 50-50 over the two actions with probability ε. Let a MENI be a MEN whose noise is due to IRT trembles. For any MENI $(Z, P, P(\varepsilon))$ and any $\varepsilon > 0$, the perturbed model has a unique limiting distribution μ^{ε} over Z, which is invariant to initial conditions (Ellison [21, sec. 2.2, p. 21]). We focus on the limit of this distribution as the trembles go to zero:

Definition 11. (Foster and Young [23]; Young [72]) The *long-run stochastically stable set* is the set of states z for which $\mu^*(z) > 0$ where $\mu^* = \lim_{\varepsilon \downarrow 0} \mu^{\varepsilon}$.

To the KMR theory we define an associated criterion:

The KMR criterion is the criterion given in (2) for the fictional beliefs

$$\Gamma_n^{\text{KMR}}(\ell) = \sum_{i=0}^{\lfloor (n-1)\ell \rfloor} W_{i,n}^{\text{KMR}} \quad \text{where} \quad W_{i,n}^{\text{KMR}} = \begin{pmatrix} n-1 \\ i \end{pmatrix} \left(\frac{1}{2}\right)^{n-1}. \tag{9}$$

In the limit as the number n of agents grows, Γ_n^{KMR} puts all of its weight on 50% of the other agents participating:

Claim 12. *In the limit as $n \to \infty$, $\Gamma_n^{KMR}(\ell)$ converges to*

$$\Gamma_{\infty}^{KMR}(\ell) = \begin{cases} 0 & \textit{if} \quad \ell < 1/2 \\ 1 & \textit{if} \quad \ell \geq 1/2. \end{cases} \tag{10}$$

In this limit, the KMR criterion selects the equilibrium with the larger basin of attraction - or, equivalently, the action that is a best response to an other-agent participation rate of one half.

If π satisfies weak SC1 then, in any MENI with Darwinian dynamics, the agents' participation decision is given by (2) for the fictional beliefs Γ_n^{KMR} if the population size N is large enough:[21]

[21] Kim [50, Prop. 2, p. 211] shows this result for KMR when payoffs satisfy strategic complementarities. We generalize Kim's result to all MENIs with Darwinian dynamics in which π satisfies weak SC1. The proof shows that the size of an action's basin of attraction is determined by the sign of γ at the midpoint $z = 1/2$ and then applies Ellison's [21] radius-coradius theorem.

Theorem 13. *Let π satisfy weak SC1 on λ and let γ be the expected relative payoff function defined in (5). For each aggregate population size N, let $\left(Z^N, P^N, P^N(\varepsilon)\right)$ be a MENI whose deterministic dynamic P^N is Darwinian with respect to γ. Then if $\int_{\ell=0}^{1} \pi(\ell)\, d\Gamma_n^{KMR}(\ell)$ is positive (resp., negative), there is an $N^* < \infty$ such that for all $N > N^*$, the only element of the long-run stochastically stable set of $\left(Z^N, P^N, P^N(\varepsilon)\right)$ is $z = 1$ (resp., $z = 0$).*

5.2 Two Theories Based on the Replicator Dynamic

We now study two continuous-time models with infinitesimal agents. At each time $t \in \Re_+$, random groups of n boundedly rational agents are selected to play the participation game. Let a_t (resp., b_t) denote the measure of agents who (do not) participate at time t and let $z_t = \frac{a_t}{a_t+b_t}$ denote the proportion who participate. The the growth rate of each population is assumed to equal its expected payoff in the participation game:

$$da = a\gamma_1(z)\, dt \quad \text{and} \quad db = a\gamma_0(z)\, dt \tag{11}$$

where

$$\gamma_i(z) = \int_{\ell=0}^{1} \kappa(\ell; z)\, u(i, \ell)\, d\mu(\ell) \tag{12}$$

is an agent's gross expected utility from playing action $i = 0, 1$.[22]

5.2.1 The FY Theory

The FY theory is due to Foster and Young [23] (FY). They solve (11) for dz to obtain the deterministic replicator dynamic: $dz = z(1-z)\gamma(z)\, dt$.[23] To this they

[22]Defined in section 2, $u(i, \ell)$ is the utility from playing action $i = 0, 1$ when a proportion ℓ of one's opponents participate, where $i = 1$ (resp., $i = 0$) is interpreted as (not) participating. The formula for $\gamma_i(z)$ is obtained by replacing $\pi(\ell)$ in (5) by $u(i, \ell)$.

[23]To see this, let $f(a, b) = \frac{a}{a+b}$ whence $dz = df(a, b) = f_1 da + f_2 db$ where subscripts indicate partial derivatives. Thus, since $f_1 = \frac{b}{(a+b)^2}$ and $f_2 = -\frac{a}{(a+b)^2}$, dz equals $\frac{b}{(a+b)^2} da - \frac{a}{(a+b)^2} db$ which can be rearranged to $z(1-z)\left(\frac{da}{a} - \frac{db}{b}\right) = z(1-z)\gamma(z)\, dt$ as $\gamma(z) = \gamma_1(z) - \gamma_0(z)$ by (1) and (12).

add shocks:

$$dz = z\,(1-z)\,\gamma\,(z)\,dt + \sigma dw, \tag{13}$$

where $\sigma > 0$ is a constant and w is a Brownian motion.[24,25] We define a corresponding criterion:

The FY criterion is the criterion given by (2) for the fictional beliefs

$$\Gamma_n^{\mathrm{FY}}(\ell) = \sum_{i=0}^{\lfloor (n-1)\ell \rfloor} W_{i,n}^{\mathrm{FY}} \quad \text{where} \quad W_{i,n}^{\mathrm{FY}} = \frac{6\,(i+1)\,(n-i)}{n\,(n+1)\,(n+2)} \tag{14}$$

in the discrete case and $\Gamma_\infty^{\mathrm{FY}}(\ell) = 3\ell^2 - 2\ell^3$ in the infinitesimal case.

Theorem 14. *Assume π satisfies weak SC1 on λ. Then for $\sigma > 0$, equation (13) has an ergodic distribution. In the long run, in the limit as σ shrinks to zero, the state z spends nearly all of its time in a small neighborhood of one (resp., zero) if $\int_{\ell=0}^{1} \pi(\ell)\, d\Gamma_n^{FY}(\ell)$ is positive (resp., negative).*

Theorem 14 assumes that each agent interacts with a random set of $n-1$ other agents. It thus corresponds to the discrete case. To address the infinitesimal case, we assume now that an agent interacts with the whole population: her flow payoff is $\pi(z_t)$ rather than $\gamma(z_t)$. Equation (13) is thus replaced with

$$dz = z\,(1-z)\,\pi\,(z)\,dz + \sigma dw. \tag{15}$$

As this is the only change, a result analogous to Theorem 14 holds if we assume that π has the properties of γ that we use in the proof of that result:

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[24] FY [23] derive their criterion in two-player, two-action coordination games. Kim [50] generalizes their result to coordination games with $n \geq 2$ players, while we replace Kim's coordination game assumption with weak SC1. For further details, see section 5.2.1 of the online appendix.

[25] FY assume that z reflects at the boundary of $[\Delta, 1-\Delta]$ for small $\Delta > 0$. Relying on a different mathematical result, we let z reflect at the boundaries of $[0, 1]$. This does not affect the results: Kim's [50] n-player generalization of FY, which also restricts $[\Delta, 1-\Delta]$, finds the same selection criterion as our Theorem 14. (While Kim assumes strategic complementarities, we rely only on weak SC1.)

Corollary 15. *Let $\pi : [0,1] \to \Re$ be Lipschitz-continuous on $[0,1]$ and satisfy SC1 on $(0,1)$. Then for $\sigma > 0$, equation (15) has an ergodic distribution. In the long run, in the limit as σ shrinks to zero, the state z spends nearly all of its time in a small neighborhood of one (resp., zero) if $\int_{\ell=0}^{1} \pi(\ell)\, d\Gamma_{\infty}^{FY}(\ell)$ is positive (resp., negative).*

Moreover, the discrete criterion converges to the continuous one as n grows:

Claim 16. *For all $\ell \in [0,1]$, $\lim_{n\to\infty} \Gamma_n^{FY}(\ell) = \Gamma_{\infty}^{FY}(\ell)$.*

5.2.2 The FH Theory

While FY add shocks directly to the state z, Fudenberg and Harris [32] (FH) instead add shocks to the sizes of the populations playing the two actions. They study a sequence of variants of this two-player random matching model until they obtain an ergodic distribution. We extend this final variant to the case of $n \geq 2$ players.[26]

In their final variant, FH assume that

$$da = a\left[\gamma(z)\, dt + \sigma_a dw_a\right] + (\lambda_b b - \lambda_a a)\, dt \tag{16}$$

and

$$db = b\left[\sigma_b dw_b\right] + (\lambda_a a - \lambda_b b)\, dt \tag{17}$$

where w_a and w_b are independent Brownian motions.[27] This model differs from FY in two ways. First, there are shocks (dw_a and dw_b) to the masses (a and b, resp.) playing each action rather than to the proportion z playing action one. FH [32, Prop. 2] show that such shocks alone do not give rise to an ergodic distribution. Intuitively, the implied shocks to z are of order $z(1-z)$ which vanishes as the state z approaches zero or one. Hence, the two endpoints of the state space $[0,1]$ are absorbing. To

[26] Fudenberg and Harris [32] restrict to two-player, two-action games. We extend this variant to the case of $n \geq 2$ players under weak SC1. To our knowledge, the FH model has been solved only in the case of two players. This contrasts with the other noncooperative theories, which Kim [50] solves for $n \geq 2$ players under the assumption of strategic complementarities.

[27] The functions γ_i for $i = 0, 1$ are defined in (12).

avoid this, FH also add individual random trembles: each agent playing a (resp., b) exogenously switches to b (resp., a) according to a Poisson process with arrival rate λ_a (resp., λ_b). These trembles keep the participation rate z away from the endpoints, thus yielding an ergodic distribution.

Define the constant

$$\sigma = \sqrt{\sigma_a^2 + \sigma_b^2}. \tag{18}$$

Let w denote $\frac{\sigma_a}{\sigma} w_a - \frac{\sigma_b}{\sigma} w_b$, which is a standard Brownian motion.[28] By (16) and (17) and Ito's Lemma, the state $z = \frac{a}{a+b}$ changes according to[29]

$$dz = \alpha(z)\, dt + \beta(z)\, dw \tag{21}$$

with coefficients

$$\alpha(z) = \ z(1-z)\left[\gamma(z) + (1-z)\sigma_b^2 - z\sigma_a^2\right] + \lambda_b(1-z) - \lambda_a z \tag{22}$$

and

$$\beta(z) = z(1-z)\,\sigma. \tag{23}$$

We assume that σ_a^2, σ_b^2, λ_a, and λ_b are all constant multiples of a common param-

[28] As w_a and w_b have continuous paths and independent increments, so does w. And since w_a and w_b are independent Brownian motions, $\sigma dw = \sigma_a dw_a - \sigma_b dw_b$ is normal with mean zero and variance σ^2. Thus, w is a standard Brownian motion.

[29] To see this, define $f(a,b) = \frac{a}{a+b}$ which equals z. By Ito's Lemma in two dimensions,

$$dz = f_1 da + f_2 db + \frac{1}{2}\left[(da)^2 f_{11} + 2(da)(db) f_{12} + (db)^2 f_{22}\right], \tag{19}$$

where numbered subscripts denote partial derivatives. In the present setting, $(da)^2 = (\sigma_a a)^2 dt$, $(db)^2 = (\sigma_b b)^2 dt$, $(da)(db) = 0$, $f_1 = \frac{b}{(a+b)^2}$, $f_2 = -\frac{a}{(a+b)^2}$, $f_{11} = -\frac{2b}{(a+b)^3}$, and $f_{22} = \frac{2a}{(a+b)^3}$ and thus, using $\gamma(z) = \gamma_1(a) - \gamma_0(z)$,

$$dz = z(1-z)\left(\frac{da}{a} - \frac{db}{b}\right) + z(1-z)\left[(1-z)\sigma_b^2 - z\sigma_a^2\right] dt. \tag{20}$$

We then use (16) and (17) to substitute for da and db in equation (20) to obtain (21). One can easily see that (21) is equivalent to FH's equation (8).

eter, which we take to be σ^2 to save notation:

$$\sigma_a^2 = s_a\sigma^2,\ \sigma_b^2 = s_b\sigma^2,\ \lambda_a = l_a\sigma^2,\ \text{and}\ \lambda_b = l_b\sigma^2 \tag{24}$$

where $s_a, s_b, l_a, l_b \in \Re_{++}$ are fixed.[30]

To the FH theory we define a corresponding criterion:

The FH criterion is the criterion given by (2) for the fictional beliefs

$$\Gamma^{\text{FH}}(\ell) = \begin{cases} 1/2 & \text{if } \ell < 1 \\ 1 & \text{if } \ell = 1. \end{cases} \tag{25}$$

That is, agents who rely on the FH criterion will play a best response to the belief that either no others or all others will participate, with equal probabilities.

In the FH model, in the long run in the limit as the noise parameter σ shrinks to zero, the agents play according to the FH criterion. More precisely, let F be the long-run distribution of the state: for any $y \in [0,1]$, $F(y) = \lim_{t\to\infty} \Pr(z_t \leq y)$. For any $y \in (0,1)$, define the two quantities

$$\chi_1^y = \frac{F(y)}{1-F(y)}\ \text{and}\ \chi_2^y = \frac{1-F(1-y)}{F(1-y)}. \tag{26}$$

Intuitively, χ_1^y (resp., χ_2^y) gives the *odds* that the state lies in $[0,y]$ (resp., in $(1-y,1]$). The formal result is as follows.

Theorem 17. *Assume either that (a) agents are discrete and π satisfies weak SC1 or (b) agents are infinitesimal and π is Lipschitz-continuous on $[0,1]$ and satisfies SC1 on $(0,1)$. Then for any $y \in (0,1)$, if $\int_{\ell=0}^{1} \pi(\ell)\, d\Gamma^{FH}(\ell)$ is negative (resp., positive) then $\lim_{\sigma\downarrow 0} \chi_1^y = \infty$ (resp., $\lim_{\sigma\downarrow 0} \chi_2^y = \infty$).*

5.3 The Matsui-Matsuyama (MM) Theory

The MM theory is due to Matsui and Matsuyama [53] (MM). In this theory, the participation game is played in continuous time by randomly chosen groups of n

[30] The parameters s_a and s_b in (24) must satisfy $s_a + s_b = 1$ as $\sigma_a^2 + \sigma_b^2 = (s_a + s_b)\sigma^2$ equals σ^2 by (18).

fully rational agents, each of whom gets Poisson chances to change actions.[31] In the case of discrete agents, MM's model is as follows. As in FY and FH, at each time $t \in \Re_+$ random groups of n agents are selected from the unit interval $[0,1]$ to play the participation game. By Claim 8, an agent's flow payoff from participating is thus $\gamma(z)$ where $z \in [0,1]$ is the participation rate in the whole population.

Unlike in FY and FH, the agents are rational and forward-looking. Each agent gets chances to switch actions according to an independent Poisson process with arrival rate $p > 0$. The rate of change of the participation rate thus satisfies $\frac{dz_t}{dt} \in [-pz_t, p(1-z_t)]$ so for any given initial participation rate $z_0 = \zeta \in [0,1]$, the path $z_\cdot = (z_t)_{t \geq 0}$ of this rate must satisfy $z_t \in \left[\underline{z}_t^\zeta, \overline{z}_t^\zeta\right]$ where the lower and upper bounds are given by

$$\underline{z}_t^\zeta = \zeta e^{-pt} \quad \text{and} \quad \overline{z}_t^\zeta = 1 - (1-\zeta)\, e^{-pt}, \tag{27}$$

respectively. Let r be the rate of time preference. As agents have perfect foresight, the benefit of switching from "not participate" to "participate" at time t, if an agent expects the path $(z_s)_{s \geq 0}$ to be played, is proportional to

$$V_t^r(z_\cdot) \stackrel{d}{=} (p+r) \int_{s=t}^{\infty} e^{-(p+r)(s-t)} \gamma(z_s)\, ds \tag{28}$$

since, for each time $s \geq t$, the expected flow payoff from participating is $\gamma(z_s)$ and the chance that the agent will not receive another revision opportunity *before* time s is $e^{-p(s-t)}$. An agent with an action revision opportunity at time t will thus choose (not to) participate if (28) is positive (resp., negative). One can easily calculate that

$$\text{if } z_t = z \text{ for all } t \geq 0, \text{ then } V_t^r(z_\cdot) = \gamma(z). \tag{29}$$

MM define the following concepts.

Definition 18. A state $z \in [0,1]$ is *accessible from* $z' \in [0,1]$ if there exists an equilibrium path from z' that reaches or converges to z. The state z is *globally*

[31] MM [53] prove their result for coordination games with two players and two actions. Kim [50, Proposition 1] generalizes it to coordination games with $n \geq 2$ players and two actions. We extend it further by replacing Kim's coordination game assumption with weak SC1.

accessible if it is accessible from any $z' \in [0,1]$.

Definition 19. A state $z \in [0,1]$ is *absorbing* if there is a neighborhood U of z such that any equilibrium path originating in U must converge to z. It is *fragile* if it is not absorbing.

To the MM theory we define an associated selection criterion:[32],[33]

The Laplace criterion is the criterion given in (2) for the fictional beliefs

$$\Gamma_n^{\text{Laplace}}(\ell) = \sum_{i=0}^{\lfloor (n-1)\ell \rfloor} \frac{1}{n} \tag{30}$$

in the discrete case and

$$\Gamma_\infty^{\text{Laplace}}(\ell) = \ell \tag{31}$$

in the infinitesimal case.

Agents who rely on the Laplace criterion play a best response to the belief that all other-agent participation rates are equally likely.

Theorem 20. *Let $\pi : \lambda \to \Re$ satisfy weak SC1. In the above version of MM with random groups of n agents who play the stage game, if $\int_{\ell=0}^{1} \pi(\ell)\, d\Gamma_n^{Laplace}(\ell)$ is positive (resp., negative), there is an $r^* > 0$ such that for all $r \in (0, r^*)$, $z = 1$ (resp., $z = 0$) is absorbing and globally accessible while $z = 0$ (resp., $z = 1$) is fragile and not globally accessible.*

We can extend Theorem 20 to the case of infinitesimal agents by assuming that an agent plays against the whole population $[0,1]$. An agent's flow payoff is thus $\pi(z)$ rather than $\gamma(z)$. Accordingly, the same results hold if π has the properties of γ on which we rely in the proof of Theorem 20:

[32]Kim [50] first identified the Laplace criterion and showed that it is implied by the GG and MM theories in coordination games. An intuition for the former result appears in MS [55, pp. 61-63].

[33]The role of the floor function in (30) is to ensure that the beliefs are defined for all $\ell \in [0,1]$ and thus that the integral in (2) is well-defined.

Corollary 21. *Let $\pi : [0,1] \to \Re$ be continuous on $[0,1]$ and satisfy SC1 on $(0,1)$. In the modified version of MM with flow payoff $\pi(z)$ rather than $\gamma(z)$, if*

$$\int_{\ell=0}^{1} \pi(\ell)\, d\Gamma_{\infty}^{Laplace}(\ell)$$

is positive (resp., negative), then there is an $r^* > 0$ such that for all $r \in (0, r^*)$, $z = 1$ (resp., $z = 0$) is absorbing and globally accessible while $z = 0$ (resp., $z = 1$) is fragile and not globally accessible.

6 A Static Theory: Global Games

The Global Games (GG) theory is a static model in which the payoff function π depends not only on ℓ but also on an agent's private signal of an unobserved "fundamental" θ, such that (not) participating is strictly dominant for sufficiently high (low) signals. A contagion argument then pins down the agents' behavior for almost any fundamental.[34,35]

[34]Global games were first studied for two-player, two-action games by Carlsson and van Damme [14] and for n-player, two-action coordination games by Kim [50]. The coordination-game assumption was first relaxed by Goldstein and Pauzner [34] who showed that there is a unique Nash equilibrium under strict SC1 if signal errors are uniformly distributed. Morris and Shin [55, p. 70] then showed the existence of a unique *threshold* equilibrium for general signal errors under strict SC1. Next Szkup [66] extend this uniqueness result to uniform signal errors and a specific payoff function that satisfies only weak SC1. Finally, we extend the result to general signal errors and a general payoff function that satisfies weak SC1.

[35]The problem of eliciting the participation of a group of agents in a global games setting has been studied in the context of asset liquidity (Plantin [60]), bailouts (Frankel [25]), bank runs (Goldstein and Pauzner [34]), debt pricing (Morris and Shin [57]), foreign direct investment (Dasgupta [18]), IMF interventions (Morris and Shin [58]), investment subsidies (Sákovics and Steiner [63]), platform competition (Argenziano [4]; Guimaraes and Pereira [38]; Jullien and Pavan [47]), regime change (Edmond [20]), and monopoly pricing (Frankel [24]). Other applications of global games include international contagion (Goldstein and Pauzner [35]), currency crises and market crashes (Morris and Shin [54, 56]), information acquisition (Yang [71]), investment cycles (Chamley [16]; Oyama [59]), merger waves (Toxvaerd [68]), neighborhood change (Frankel and Pauzner [30]), regime change (Angeletos, Hellwig, and Pavan [1, 2]; Szkup and Trevino [67]), search-driven business cycles (Burdzy and Frankel [13]), and sectoral choice (Frankel and Pauzner [29]). Experimental support appears in Heinemann, Nagel, and Ockenfels [42, 43], while theoretical limitations are studied in Angeletos, Hellwig, and Pavan [1], Angeletos and Werning [3], Chassang [17], Hellwig, Mukherji, and Tsyvinski [44], Morris and Shin [57], and Weinstein and Yildiz [69]. Strategic substitutes are studied in Harrison and Jara-Moroni [41] and Hoffman and Sabarwal [45], who extend the uniqueness result,

Following MS [55, Lemma 2.3, p. 70], we assume there is a random, unobserved state θ that is uniformly distributed on the whole real line.[36,37] Agents may be either discrete or infinitesimal. Each agent i sees a noisy signal $x_i = \theta + \sigma\varepsilon_i$ of the state θ, where $\sigma > 0$ is a scalar. An agent's payoff from participating is a function $\pi(\ell, x_i)$ of the other-agent participation rate ℓ and the agent's private signal x_i.[38] The idiosyncratic terms ε_i are independent of each other and of θ and have a continuous density f with full support on $\Re$ and corresponding distribution function F. We assume moreover that f satisfies

MLRP For all $x_H > x_L$ and $y_H > y_L$ in $\Re$,

$$f(s_H - z_H) f(s_L - z_L) \geq f(s_H - z_L) f(s_L - z_H).$$

An example with these properties is a normal distribution (with any mean and variance). MLRP appears as assumption A7 in MS [55, p. 69].

Fix an agent i. Suppose each agent $j \neq i$ participates (resp., does not participate) if her signal $x_j = \theta + \sigma\varepsilon_j$ exceeds (resp., is less than) some fixed threshold $k \in$

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and Karp, Lee, and Mason [49], who show that a threshold equilibrium may not exist if weak SC1 fails.

[36]Such an "improper prior" can be seen as the limit, e.g., of a normal distribution as the variance grows without bound. The assumption of an improper prior greatly simplifies the analysis but is not essential to the results.

[37]In some applications (e.g., Frankel [27]), a principal designs a scheme to induce the agents to participate. The principal knows the state and thus can predict the agents' response. We can obtain this feature by assuming that a public signal $\theta + \eta\varepsilon$ of the state is first observed, where $\eta > 0$ is a scalar and ε is noise. The principal chooses her scheme, after which each agent i sees a private signal $x_i = \theta + \sigma\varepsilon_i$ of the state. Taking the private signal error to zero ($\sigma \to 0$), one obtains a unique prediction for the agents' behavior. If we then take the public signal error to zero ($\eta \to 0$), the principal can estimate the state with arbitrarily high precision; she can thus predict how the agents will respond to any scheme. This order of limits was previously studied in Morris and Shin [58].

[38]Frankel, Morris, and Pauzner [31, pp. 23 ff.] (FMP) give conditions under which, if π satisfies *strategic complementarities*, then the asymptotic (as $\sigma \to 0$) behavior of the model studied here coincides with that of a variant in which an agent's payoff is a function $\pi(\ell, \theta)$ not of her signal x_i but rather of the state θ. While we expect that their result holds also under weak SC1, the verification needed would be beyond the scope of this paper for two reasons. First, the Laplace criterion is just one of six criteria that we study and finds support also in the MM theory (section 5.3). Second, real-world payoffs are plausibly heterogeneous, which is better captured by a model such as ours in which π depends partly on an idiosyncratic shock ε_i.

$\Re \cup \{\pm\infty\}$. Given θ, the probability that agent j participates is then

$$\Pr(\theta + \sigma\varepsilon_j > k|\theta) = \Pr\left(\varepsilon_j > \frac{k-\theta}{\sigma}\middle|\theta\right) = 1 - F\left(\frac{k-\theta}{\sigma}\right) \tag{32}$$

We now turn separately to the cases of discrete and infinitesimal agents.

1. Discrete agents: $R = \lambda$. By (32), the probability (given θ) that a proportion $\ell \in \lambda$ of agent i's opponents participate is $\kappa\left(\ell; 1 - F\left(\frac{k-\theta}{\sigma}\right)\right)$ where κ is defined in (3). The density of the state θ at any realization θ_0 given the signal x_i is $\frac{1}{\sigma} f\left(\frac{x_i - \theta_0}{\sigma}\right)$. Thus, agent i's payoff from participating when her realized signal is x (and all others play according to the threshold k) is

$$\pi_\sigma^n(x,k) = \int_{\theta=-\infty}^{\infty} \frac{1}{\sigma} f\left(\frac{x-\theta}{\sigma}\right) \sum_{\ell \in \lambda} \kappa\left(\ell; 1 - F\left(\frac{k-\theta}{\sigma}\right)\right) \pi(\ell, x)\, d\theta. \tag{33}$$

2. Infinitesimal agents: agent i's payoff from participating when her signal realization is x is

$$\pi_\sigma^\infty(x,k) = \int_{\theta=-\infty}^{\infty} \frac{1}{\sigma} f\left(\frac{x-\theta}{\sigma}\right) \pi\left(1 - F\left(\frac{k-\theta}{\sigma}\right), x\right) d\theta \tag{34}$$

   as the other-agent participation rate ℓ equals the probability $F\left(\frac{k-\theta}{\sigma}\right)$ that a given opponent participates by the law of large numbers.

A threshold equilibrium consists of a finite threshold k^* such that if an agent i believes that each opponent j will (not) participate if her signal x_j exceeds (is less than) k^*, then it is a best response for i (not) to participate if her signal exceeds (resp., is less than) k^*. More precisely, let $\pi_\sigma^*(x,k)$ denote $\pi_\sigma^\infty(x,k)$ in the infinitesimal case and $\pi_\sigma^n(x,k)$ in the discrete case. We define:[39]

Definition 22. A *Threshold Equilibrium* consists of a threshold $k^* \in \Re \cup \{\pm\infty\}$ such that $\pi_\sigma^*(x,k^*) \gtreqless 0$ for all $x \gtreqless k^*$ in $\Re$.

Theorem 23 below gives sufficient conditions for the game to have a unique thresh-

[39] The interpretation of $k^* = +\infty$ (resp., $k^* = -\infty$) is that the players never (resp., always) participate, regardless of their signals.

old equilibrium, with a finite threshold k^*.[40] The conditions run roughly as follows. A1 states that π is bounded and satisfies weak SC1. A4 states that there are dominance regions: for a sufficiently high (low) signal, (not) participating is strictly dominant. A2 and A3 are technical continuity and boundedness properties, while A5 restates our above assumptions on the noise distribution.

Theorem 23. *Let R and $\pi(\ell, x)$ denote λ and $\pi^n(\ell, x)$, resp., in the discrete case, and $[0,1]$ and $\pi^\infty(\ell, x)$, resp., in the infinitesimal case. Assume:*

A1. *Single Crossing: for each $x \in \Re$, $\pi(\ell, x)$ satisfies weak SC1 in $\ell \in R$ and $\sup_{\ell \in R} |\pi(\ell, x)|$ is finite;*

A2. *Continuity: $\pi(\ell, x)$ is continuous in $x \in \Re$ uniformly in $\ell \in R$;*[41]

A3. *State Monotonicity: $\pi(\ell, x)$ is increasing in x and bounded in $\ell \in R$ for each $x \in \Re$;*

A4. *Dominance Regions: there are $\underline{x} < \overline{x}$ both in $\Re$ such that, for all $\ell \in R$, $\pi(\ell, \underline{x}) < 0 < \pi(\ell, \overline{x})$; and*

A5. *MLRP and Full Support: the noise density f is positive and continuous on $\Re$ and satisfies MLRP.*

Then the game has a unique threshold equilibrium, whose threshold k^* is finite and is the unique $k \in \Re$ that satisfies $\pi^*_\sigma(k, k) = 0$. In this equilibrium, an agent with signal x (does not) participate if $\int_{\ell=0}^{1} \pi(\ell, x)\, d\Gamma^{Laplace}(\ell)$ is positive (negative).

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[40] By focusing on threshold equilibria, we enable applications that satisfy only weak SC1. Other global games papers that restrict to threshold equilibria include Angeletos, Hellwig, and Pavan [2], Mathevet and Steiner [80], and Morris and Shin [56]. GP [34, pp. 1325-6] give some rationales for this restriction. It can be omitted under strategic complementarities, as shown in FMP [31] and Kim [50].

[41] More precisely: for any $\varepsilon > 0$ there is a $\delta > 0$ such that, for any $\ell \in R$ and $x, x' \in \Re$ satisfying $|x - x'| < \delta$, $|\pi(\ell, x) - \pi(\ell, x')| < \varepsilon$.

To remain faithful to the global games literature we have assumed that each agent plays against all of the other agents rather than against a random sample as in KMR, FY, FH, and MM. Yet the expected payoff function in this setting is still a Bernoulli mixture over the realized payoff function π, so that weak SC1 suffices (assumption A1). Intuitively, an agent knows her opponents' identities but not their signals. Hence, each opponent's action is again the outcome of a random trial, where the success probability is the chance that the opponent's signal exceeds the participation threshold.

7 Summary of Results

The preceding results are summarized in Table 1. With discrete agents, weak SC1 is a sufficient condition for each theory to yield the associated criterion. This condition is typically easy to verify.[42] The six fictional beliefs appear in Figure 2.

		Sufficient Conditions for Theory to Imply Criterion		
Theory	Criterion	Discrete Agents	Infinitesimal Agents	Prior Art
A. Heuristic Arguments				
Pareto	Pareto	Weak SC1	Weak SC1	SC
UI	UI	Weak SC1	Weak SC1	SC
B. Rational-Player Models				
MM	Laplace	Weak SC1	Continuous on $[0,1]$ & SC1 on $(0,1)$	SC
GG	Laplace	Weak SC1	Weak SC1	See §1
C. Evolutionary Models				
KMR	KMR	Weak SC1	Theory requires discrete agents	SC
FY	FY	Weak SC1	Lipschitz on $[0,1]$ & SC1 on $(0,1)$	SC
FH	FH	Weak SC1	Lipschitz on $[0,1]$ & SC1 on $(0,1)$	SC, $n=2$

Table 1: Summary of Results. "Prior Art": sufficient conditions identified by the prior literature. "SC": strategic complementarities. "Lipschitz": Lipschitz-continuous.

[42]See, e.g., Frankel [26, 27].

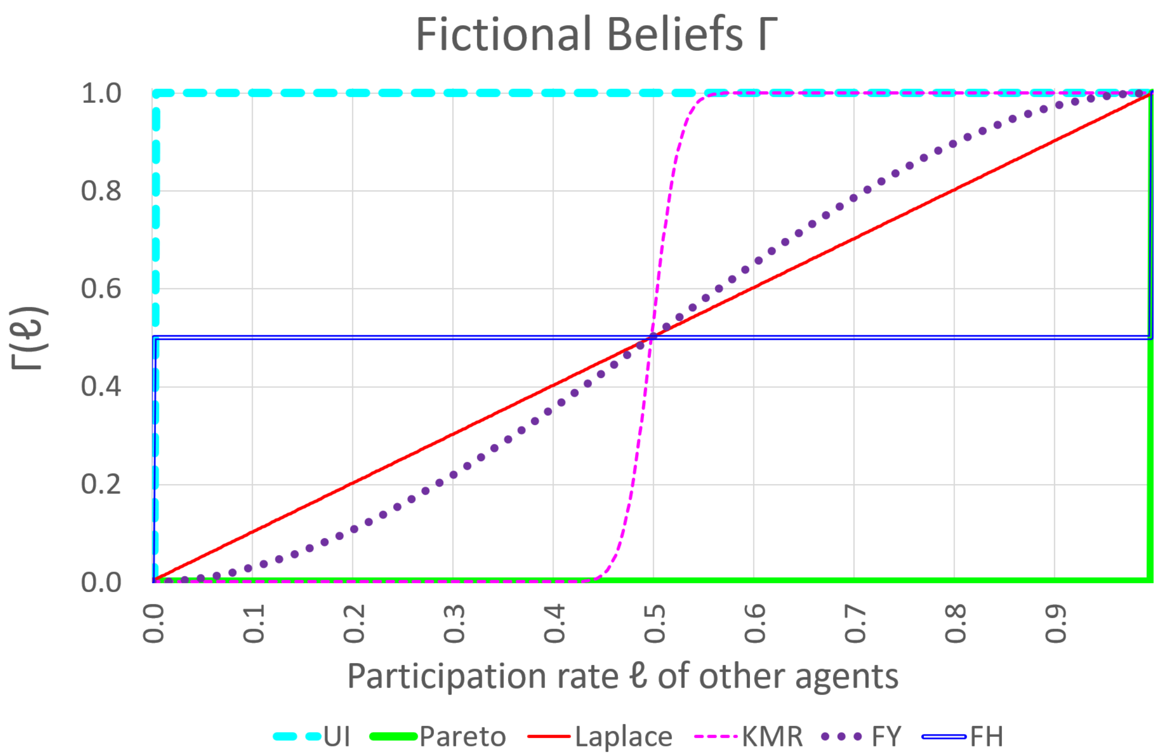

Figure 2: Fictional beliefs for each of the six criteria; $n = 500$ agents. The graph depicts the probability $\Gamma(\ell)$ that the proportion of others who participate is at most $\ell \in [0, 1]$ according to the fictional beliefs Γ associated with the UI, Pareto, Laplace, KMR, FY, and FH criteria.

7.1 Evolutionary Criteria: Discussion

The evolutionary criteria differ starkly. The fictional beliefs in KMR put most of their weight on participation rates near 50% and converge to a step function at this value as $n \to \infty$. The fictional beliefs in the FY criterion are more dispersed but, like KMR, are higher near 50%. Finally, in the FH criterion, the fictional beliefs put equal weight on participation rates of 0% and 100% and no weight on intermediate participation rates.

What explains these differences? In all three settings, the agents play a game repeatedly over time. Actions that yielded higher payoffs in the recent past are more likely to be played today. This means that if the participation rate z is close enough to zero (resp., one), it will fall (resp., rise): there are two long-run equilibria which, in our setting, are "all-participate" and "none-participate".

In order to select between these two equilibria, the three theories introduce shocks in different ways. In KMR, in each period t, every agent trembles (chooses the suboptimal action) with some small probability. Starting in a situation in which all agents play the same action, the size of that action's basin of attraction is proportional to the number of simultaneous trembles that must occur for the *other* action to become optimal - and thus, given Darwinian dynamics, for a transition to the other action to occur. But the action with the larger basin of attraction is the action that is a best response to a 50-50 action distribution. For this reason, players will eventually settle on the action that is a best response to the fictional belief that half of the other players participate.[43]

In contrast, rather than jumping as in KMR, the participation rate z in FY and FH is a continuous-time process that drifts towards the action with the higher current payoff: it has the form $dz = \alpha(z)\,dt + \beta(z)\,dw$ where w is a Brownian motion, $\beta(z)$ measures the size of the shocks, and $\alpha(z)$ measures the drift which, in turn, depends positively on the expected payoff from participating at z.[44] To transition from 0 to 1, the action distribution z must pass through every value in $(0,1)$. In this setting, the drift at all states matters. However, it matters more where the shocks are smaller:[45] the weight on the drift at a given state z is proportional to the inverse $1/\beta^2(z)$ of

[43]For fixed n, in the limit as $\varepsilon \to 0$, Γ converges not to a step function at $\ell = 1/2$ but rather to the smooth shape shown in Figure 2. Intuitively, if $n-1$ players are chosen at random from an infinite population that is evenly split between the two actions ($z = 1/2$) then the probability that exactly i of them participate is just the binomial coefficient $W_{i,n}^{\mathrm{KMR}}$ defined in (9). Hence, "participate" has a larger basin of attraction if an agent would choose it under the belief that, for each $i = 0, ..., n-1$, the probability is $W_{i,n}^{\mathrm{KMR}}$ that exactly i of her $n-1$ opponents participate.

[44]The drift $\alpha(z)$ equals $z(1-z)\gamma(z)$ where $\gamma(z)$ is the expected flow payoff from participating against $n-1$ opponents, each of whom has an independent chance z of participating. For more details, see sections 3 and 5.2 of the online appendix.

[45]Similarly, Beggs [7] finds that an equilibrium is more likely to be selected if the shocks are smaller in its basin of attraction. More generally, Bergin and Lipman [8] and Binmore and Samuelson [10] show that equilibrium selection depends on the sizes of the shocks at different states.

the squared shock at that state.[46] In FY, the diffusion term $\beta(z)$ is a constant so the criterion is a simple average of the drift over all states. This corresponds to the diffused fictional beliefs Γ^{FY} depicted in Figure 2. In FH, in contrast, the weight $1/\beta^2(z)$ goes to infinity as the participation rate z approaches either zero or one. This gives rise to the fictional beliefs Γ^{FH} in Figure 2, which divide their weight equally between these two extreme rates.[47]

Much of the evolutionary literature has focused on the robustness of a particular selection criterion to changes in the noise structure. For instance, Sandholm [64, Thm. 7.14] shows that in a large class of n-player two-action games, such robustness requires a payoff condition that is stronger even than strategic complementarities. Our approach is different: we weaken the payoff conditions as much as possible and show that the resulting criteria, while heterogeneous, have a simple common form.

8 Mechanism Design

An intended application of our findings is to embed the participation game in a larger setting in which a principal designs a scheme to induce the agents to participate. For the principal to have an optimal scheme, the set of schemes that induce participation must typically be closed. However, this is not so under (2) as the inequalities are strict. We now develop a toolkit to address this issue.

We assume the principal can either stay out, getting some fixed payoff U_0, or propose a scheme s from a nonempty set Σ that is compact with respect to some metric μ.[48] Fix an agent size (discrete or infinitesimal) and let R be the set of feasible

[46] See Lemmas 43 and 44 in section 10 of the online appendix.

[47] Blume [12] studies a discrete-time model in which groups of $n \geq 2$ players are randomly matched to play a two-action coordination game. Relative to our version of KMR, Blume makes stronger assumptions on payoffs but imposes weaker conditions on noise. He shows that either the FH criterion [12, Thms. 3 & 4] or the KMR criterion [12, Thm. 5] may emerge, depending on the form of the noise.

[48] See Appendix A for a review of metric spaces and compactness.

other-agent participation rates.[49] Let $\pi_s : R \to \Re$ be the agents' payoff function from participating in s. The agents' decision is thus governed by (2) with $\pi = \pi_s$, which yields three cases. If $\Xi(\pi_s)$ is positive, all agents participate; we let U_s denote the principal's payoff in this case. If $\Xi(\pi_s)$ is negative, no agents participate; we assume this is dominated by staying out.[50]

Finally, if $\Xi(\pi_s)$ is zero, the agents' response is indeterminate. We assume the principal will not propose such a scheme. As the justification for this assumption is somewhat complicated, we defer it to Appendix B.

Assume that the maps $s \to U_s$ and, for all $\ell \in R$, $s \to \pi_s(\ell)$ are continuous with respect to μ. Let

$$\varphi_\Gamma(s) = \int_{\ell=0}^{1} \pi_s(\ell)\, d\Gamma(\ell) \tag{35}$$

denote the integral in (2) associated with the payoff function π_s.

Claim 24. *The map $s \to \varphi_\Gamma(s)$ is continuous in s with respect to μ.*

Proof. Online appendix, section 10. □

Assume that for every scheme s in Σ, the payoff function π_s satisfies the sufficient conditions in Table 1 for (2) to hold.[51] Let

$$O = \{s \in \Sigma : \varphi_\Gamma(s) > 0\} \tag{36}$$

denote the set of *successful* schemes in Σ: those that induce the agents to participate under (2). As noted, we assume the principal will propose a successful scheme or no scheme at all. But as the set O is not closed, the principal may not have an optimal

[49] R equals $\lambda = \left\{\frac{i}{n-1} : i = 0, ..., n-1\right\}$ if agents are discrete and $[0,1]$ if infinitesimal.

[50] In practice, devising and marketing a scheme takes time and rejection is likely to damage a principal's reputation.

[51] These conditions depend on the researcher's chosen agent size: discrete or infinitesimal. Sometimes the issuer has no choice; e.g., if $\pi_s(\ell)$ is discontinuous in ℓ for some schemes s, she must assume discrete agents. In more flexible applications, her choice might be guided instead by the prior literature, notational simplicity, or - if she allows both agent sizes - a desire for generality. For a variety of such applications, see Frankel [27].

successful scheme. Instead, we look for an approximate optimum, which we define as follows.

Definition 25. A scheme $s^* \in \Sigma$ is *approximately optimal in* Σ if (a) there is no successful scheme $s' \in \Sigma$ that satisfies $U_{s'} > U_{s^*}$ and (b) for any $\varepsilon > 0$ there is a successful scheme $s' \in \Sigma$ within ε of s^*, such that $|U_{s^*} - U_{s'}| < \varepsilon$.

By part (a), an approximately optimal scheme s^* provides a tight upper bound on the principal's payoff from a successful scheme. By part (b), there are successful schemes near s^* that give the principal a payoff near this upper bound. Finally, an approximately optimal scheme always exists by the following result. Let

$$\overline{O} = \bigcap \{\text{closed } S' \subseteq \Sigma : O \in S'\} \tag{37}$$

denote the closure of O.

Claim 26. *Assume O is nonempty. Then there exists a solution to* $\max_{s \in \overline{O}} U_s$. *Moreover, s^* is approximately optimal in Σ if and only if it solves* $\max_{s \in \overline{O}} U_s$.

Proof. Online appendix, section 10. □

However, solving $\max_{s \in \overline{O}} U_s$ is *not* equivalent to maximizing U_s subject to $\varphi_\Gamma(s) \geq 0$ as there may be schemes s satisfying $\varphi_\Gamma(s) = 0$ that are not near any successful schemes.[52,53] Rather, the following search procedure should be used. Steps (a) through (d) reiterate the above groundwork, while (e) through (h) are new.

[52] A scheme s is *not near any successful schemes* if it is not a limit point of schemes in O: if, for some $\delta > 0$, there is no scheme s' in O for which $\mu(s, s') < \delta$.

[53] For instance, let Γ put all of its weight on $\ell = 1/2$. Let the set Σ consist of two schemes s_1 and s_2 with payoff functions $\pi_{s_1}(\ell) = \begin{cases} 0 & \text{if } \ell < 1/4 \\ 1 & \text{if } \ell \geq 1/4 \end{cases}$ and $\pi_{s_2}(\ell) = \begin{cases} 0 & \text{if } \ell < 3/4 \\ 1 & \text{if } \ell \geq 3/4 \end{cases}$. Assume the principal prefers s_2 to s_1 if all agents participate, but prefers s_1 to staying out: $U_{s_2} > U_{s_1} > U_0$. Then s_2 maximizes U_s while satisfying $\varphi_\Gamma(s_2) = \pi_{s_2}(1/2) \geq 0$. However, s_2 is not approximately optimal since s_1 is the only successful scheme and it is not close to s_2. (If it were, $\pi_{s_1}(\ell)$ would have to be close to $\pi_{s_2}(\ell)$ for all ℓ by continuity.)

Heuristic Search Procedure (HSP) (a) Specify an agent type (discrete or infinitesimal) and let $R \in \{\lambda, [0,1]\}$ be the set of feasible other-agent participation rates. (b) Specify a nonempty set Σ of schemes and, for each scheme s in Σ, a payoff function $\pi_s : R \to \Re$ for the agents as well as the payoff $U_s \in \Re$ that the principal receives if all agents participate. (c) Specify a metric μ on Σ and verify that the set Σ is compact and the maps $s \to U_s$ and, for all $\ell \in R$, $s \to \pi_s(\ell)$ are continuous with respect to μ. (d) Show that for any scheme $s \in \Sigma$, π_s satisfies the sufficient conditions in Table 1 for the chosen agent type (discrete or infinitesimal). (e) If the set O of successful schemes is empty, abort the procedure.[54] (f) Let Σ' be the result of removing from Σ an arbitrary (and possibly empty) set of schemes that are not near any successful schemes.[55] (g) Find a scheme s^* that maximizes U_s on Σ' subject to

$$\varphi_\Gamma(s) \geq 0. \tag{38}$$

(h) Show that for every $\delta > 0$ there is a successful scheme $s' \in \Sigma$ that is within δ of s^*.

Claim 27. *Assume steps (a) through (d) of HSP are satisfied. (A) If a scheme $s^* \in \Sigma$ solves steps (e) through (h) of HSP, then it is approximately optimal in Σ. (B) If s^* is approximately optimal in Σ, then there is a way to delete schemes in step (f) such that s^* satisfies steps (e), (g), and (h).*

Proof. Online appendix, section 10. □

Frankel [27] uses HSP to solve three different variants of a security design model. For each variant, he shows that (g) implies (h) and then solves (g) using standard constrained optimization methods. In one variant, schemes must be deleted in step (f) for (g) to imply (h). Some variants posit discrete agents and others infinitesimal.

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[54] If O is empty, so is its closure $\overline{O}$: by Claim 26, there is no approximately optimal scheme.

[55] See n. 52.

A reader who wishes to apply HSP may thus find Frankel [27] useful.[56]

9 Concluding Remarks

In many applied settings, a group of agents choose whether or not to participate in a joint activity that is worthwhile only if the overall participation rate is sufficiently high. Examples include joining a platform, investing in a project, rebelling against a regime, attacking a currency, and leaving funds in a bank. Such games have multiple equilibria: both all-participate and none-participate are self-fulfilling prophecies. To make predictions, one thus needs a theory of equilibrium selection.

We study seven well-known selection theories from the literature: two based on heuristics, two that assume rational players, and three evolutionary models. We show that these theories give rise to six distinct selection criteria, all of which have same parsimonious form: an agent plays a best response to the belief that the proportion of others who participate has some fixed distribution that depends on the theory. This distribution is common across the agents and does not depend on the game's payoffs.

In deriving these implications, we assume only a weak single crossing property[57] in contrast with the prior literature, which has relied largely on strategic complementarities. This advance can make it easier to design optimal mechanisms, as well as to explain the use of existing mechanisms that violate the stronger conditions.[58]

In many settings, a principal devises a scheme to induce a set of agents to participate in some activity. To find an optimal such scheme, a researcher may need to permit payoffs that satisfy a single crossing property but not strategic complementarities. To facilitate such applications, we develop an algorithm: the Heuristic Search

[56] Another application appears below in Appendix B.

[57] This holds for discrete agents. For the infinitesimal case, see Table 1.

[58] An alternative is to craft taxes and subsidies that ensure that payoffs satisfy strategic complementarities as in Mathevet [52]. However, that approach is not as well suited to explaining existing policies.

Procedure (HSP). Frankel [27] illustrates the application of HSP in three security design settings, two of which violate strategic complementarities in a natural way.

A Compact Metric Spaces

In HSP(c) one must show that the set Σ of feasible schemes is compact with respect to some metric μ - or, equivalently, that Σ is a compact metric space with metric μ. We now briefly review some concepts that may prove useful in establishing this property.

A *metric space* is a set Σ together with a metric μ: a real-valued, nonnegative, and symmetric function on Σ^2 such that (i) for all s, s' in Σ, $\mu(s, s') > (=) \, 0$ if $s \neq (=) \, s'$ and (ii) for all s, s', s'' in Σ, $\mu(s, s'') \leq \mu(s, s') + \mu(s', s'')$ (Royden and Fitzpatrick [62, p. 183]) (RF). A metric space Σ is compact if and only if it is *sequentially compact*, which means that every sequence $(s_i)_{i=1}^{\infty}$ in Σ has a subsequence that converges to a point in Σ (RF [62, p. 199]). If, moreover, Σ is a subset of Euclidean space, then Σ is compact if and only if it is closed and bounded (RF [62, p. 200]).

B Indeterminate Schemes

Let an *indeterminate scheme* be a scheme s for which the selection criterion (2) provides no prediction: $\Xi(\pi_s)$ equals zero. In section 8, we assume that the principal will not propose such a scheme. We now justify this assumption.

There are two types of indeterminate schemes: (A) those that are near successful schemes and (B) those that are not. While we do not let an issuer propose a type-A scheme, we do not ignore them. To the contrary: they play a key role in our analysis. In particular, all type-A schemes are in the closure of the set of successful schemes and thus, by Claims 26 and 27, may be selected by HSP. Intuitively, a type-A scheme s can always be "sweetened" by passing to a nearby successful scheme s', thus inducing all agents to participate. It thus is a good approximation to schemes that the principal might select.

In contrast, a type-B scheme s cannot be sweetened to induce full participation. Accordingly, there can exist type-B schemes that would give the issuer a very high payoff *were* the agents to participate, but that are weakly dominated for the agents and thus unlikely to be accepted. Letting the principal choose such schemes can easily lead to absurd results.

To illustrate, suppose there are n agents, each with wealth $w > 0$. If an agent participates in the principal's activity, she gets a private benefit $f + b\ell$ where $\ell \in \lambda$ is the other-agent participation rate. The constants f and b are positive but much smaller than w. The principal devises a fee schedule $\tau_s : \lambda \to [0, w]$, where an agent pays $\tau_s(\ell)$ to participate.[59] An agent's payoff function is thus

$$\pi_s(\ell) = f + b\ell - \tau_s(\ell). \tag{39}$$

The principal's payoff is simply her fee revenue: if all agents participate, she gets

$$U_s = n\tau_s(1). \tag{40}$$

We now carry out the steps of HSP. For HSP(a), the agents are discrete so $R = \lambda$. For HSP(b), we use the payoff functions π_s and U_s given in (39) and (40) and define the set Σ of feasible schemes to be the set of all fee schedules $(\tau_s(\ell))_{\ell\in\lambda}$ in $[0, w]^n$ satisfying:

$$\text{For all } \ell' > \ell \text{ both in } \lambda, \text{ either } \pi_s(\ell) \leq 0 \text{ or } \pi_s(\ell') \geq 0. \tag{41}$$

For HSP(c), we use the supremum metric $\mu(s, s') = \max_{\ell\in\lambda} |\tau_s(\ell) - \tau_s(\ell')|$. As (41) imposes only weak inequalities, Σ is a closed and bounded subset of Euclidean space and thus is compact (Appendix A). The maps $s \to U_s$ and, for all $\ell \in R$, $s \to \pi_s(\ell)$ are also clearly continuous with respect to μ. For HSP(d), weak SC1 holds by (41).

[59] If a total of $i \in \{1, ..., n\}$ agents participate, then each such agent faces the same other-agent participation rate $\ell = \frac{i-1}{n-1}$ and thus pays the same fee $\tau_s(\ell)$.

As for HSP(e), the zero scheme $\tau_s(\ell) \equiv 0$ is successful as it yields the payoff function $\pi_s(\ell) = f + b\ell$ which is positive for all ℓ. For HSP(f), we remove no schemes.

We turn now to HSP(g). To obtain an upper bound on the principal's payoff under any fictional beliefs, we solve HSP(g) under the Pareto criterion: the best criterion for the principal. The integral $\varphi_{\Gamma^{\text{Pareto}}}(s)$ in (35) now equals $\pi_s(1) = f + b - \tau_s(1)$. Thus, HSP(g) is solved by the scheme s^* whose fee $\tau_{s^*}(\ell)$ identically equals the full-participation private benefit $f + b$.[60] By (40), an upper bound on the principal's payoff from any successful scheme, under any fictional beliefs, is thus the sum of the agents' full-participation private benefits:

$$U_s \leq n(f + b). \tag{42}$$

We now show that a much higher payoff U_s emerges if type-B schemes are allowed, which is equivalent to omitting step (h) from HSP. Moreover, we prove this result under the UI criterion: the worst criterion for the principal. Consider the scheme $s^\dagger$ given by

$$\tau_{s^\dagger}(\ell) = \begin{cases} f & \text{if } \ell = 0; \\ w & \text{else.} \end{cases}$$

The resulting payoff function is

$$\pi_{s^\dagger}(\ell) = \begin{cases} 0 & \text{if } \ell = 0; \\ f + b\ell - w \ll 0 & \text{else} \end{cases} \tag{43}$$

which satisfies weak SC1 as it is never positive: $s^\dagger$ is in Σ. Moreover, an agent who participates gets zero if no others participate: $s^\dagger$ satisfies the constraint (38) of HSP(g). Since, by (40), the principal's payoff $U_{s^\dagger}$ under full participation equals the agents' aggregate wealth nw, there is no better scheme for her: $s^\dagger$ solves HSP(g). And since $w \gg f + b$, the payoff $U_{s^\dagger}$ far exceeds the upper bound (42) on the principal's

[60] The resulting payoff function $\pi_{s^*}(\ell) = -b(1 - \ell)$ is increasing in ℓ so it satisfies (41): s^* is in Σ. Moreover, one can obtain a successful scheme by reducing the fee to $\tau_s(\ell) = f + b - \delta$ for any small $\delta > 0$, so s^* satisfies HSP(h) as well.

payoff from any acceptable scheme.

However, $s^{\dagger}$ lies far away from any successful scheme and thus fails HSP(h) badly. Why? By (36), a successful scheme s' must satisfy $\varphi_{\Gamma^{\mathrm{UI}}}(s') = \pi_{s'}(0) > 0$. But then by weak SC1, its payoff function must be nonnegative and so, by (43), must vastly exceed $\pi_{s^{\dagger}}(\ell)$ for $\ell > 0$. The scheme $s^{\dagger}$ is also implausible: no reasonable agent would agree to a scheme that leaves her indifferent if noone else participates but leaves her penniless otherwise. We conclude that to avoid absurd results, one must assume that the principal will not propose an indeterminate scheme.

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