Optimal Public Debt with Life Cycle Motives*

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Abstract

In a seminal paper, Aiyagari and McGrattan (1998) find that in a standard incomplete markets model with infinitely lived agents it is optimal for the U.S. government to have a large amount of public debt. Debt is optimal because it induces a higher interest rate, which encourages more household savings and better self-insurance. This paper revisits their result in a life cycle model only to find that public debt's insurance enhancing mechanism is severely limited. While a higher interest rate encourages higher average savings in both models, the benefits vary. In a life cycle model, agents enter the economy with no savings but must accumulate the higher level of savings throughout their lifetime, thereby eliminating some of the benefits. In contrast, infinitely lived agents do not accumulate savings over a lifetime and, thus, simply enjoy the benefit of the higher average savings *ex ante*. Overall, we find that while optimal debt is equal to 24% of output in the infinitely lived agent model, when a life cycle is introduced it is optimal for the government to hold savings equal to 59% of output. Not accounting for life cycle features when computing optimal policy reduces welfare by nearly one-half percent of expected lifetime consumption.

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1 Introduction

Government debt is pervasive across economies. In the decades preceding the Great Recession, debt-to-GDP ratios in advanced economies averaged over 40 percent. Motivated by its prevalence, this paper examines the optimality of public debt with a specific focus on the U.S. economy.

In their seminal work, Aiyagari and McGrattan (1998) find that it is optimal for the government to be a net borrower, with a large amount of public debt (on the order of magnitude of two-thirds the size of GDP), in a standard incomplete markets model with infinitely lived agents. Government debt is optimal because it crowds out the stock of productive capital and leads to a higher interest rate that encourages households to save more. As a result, households are better self-insured against idiosyncratic labor earnings risk and therefore less likely to face binding borrowing constraints. While savings behavior is central to public debt being optimal, introducing a life cycle can change savings behavior and, therefore, affect optimal debt policy. However, the potential effect of the life cycle on optimal debt policy has largely been ignored.

This paper characterizes the effect of a life cycle on optimal public debt and evaluates the mechanisms by which a life cycle affects optimal policy. In order to determine the effect of the life cycle, we compute optimal policy in two model economies. The first is the standard incomplete markets model with infinitely lived agents. The second is the life cycle model, which is similar to the infinitely lived agent model except that it introduces finite lifespans, mortality risk, an age-dependent wage profile, endogenous retirement and a Social Security program. We find that the optimal policies are strikingly different between the two models. In the infinitely lived agent model it is optimal for the government to be a net borrower with debt equal to 24 percent of output. In contrast, in the life cycle model, we find that it is no longer optimal for the government to have public debt. Instead, it is optimal for the government to be a net saver with public savings equal to 59 percent of output.

Our results demonstrate that studying optimal policy in an infinitely lived agent model, which abstracts from the realism of a life cycle in favor of computational tractability, is not without loss of generality. Not only is the optimal policy quite different when one ignores life cycle features, but the welfare consequences of ignoring them are economically significant. In the life cycle model, we find that if a government implemented the 24 percent debt-to-output policy that is optimal in the infinitely lived agent model, instead of the actual optimal policy of public savings equal to 59 percent of output, then life cycle agents would be worse off by nearly 0.5 percent of expected lifetime consumption.

The difference in optimal policies between models can be explained by life cycle agents' special progression through distinct phases over their lifetimes. Specifically, life cycle model agents begin their life with little or no savings and enter an *accumulation phase* in which they accumulate a precautionary stock of savings to insure against labor earnings risk and finance their post-retirement consumption. In order to facilitate accumulation, agents work more and consume less on average. In middle life, agents may enter a *stationary phase* in which they have accumulated a target level of assets, around which savings fluctuates.[1] Finally, older agents enter a *deaccumulation phase* in which they decrease their savings in anticipation of death. However, in the infinitely lived agent model, agents do not experience an accumulation phase but instead experience a perpetual stationary phase.

The existence of the accumulation phase is the predominant reason for the drastically different optimal policies between the life cycle and infinitely lived agent models. In the infinitely lived agent model, higher public debt corresponds to a higher steady state level of private savings. From an ex ante perspective, the average agent has more savings and is better insured against labor earnings risk. In the life cycle model, in contrast, agents enter the economy with little or no wealth and begin accumulating savings. While a higher level of public debt might encourage life cycle agents to hold more savings during their lifetime, the fact that agents must accumulate this savings stock mitigates the welfare benefits from public debt.

This paper is related to an established literature that uses the standard incomplete market model with infinitely lived agents, originally developed in Bewley (1986), İmrohoroğlu (1989), Huggett (1993) and Aiyagari (1994), to study the optimal level of steady state government debt. In contrast to this paper, previous work has almost exclusively studied infinitely lived agent models and tends to find that public debt is optimal. Aiyagari and McGrattan (1998) is the

[1] In life cycle models where agents live for a short enough span, agents sometimes transition directly from the accumulation phase to the deaccumulation phase skipping this stationary phase. We generally find this to be the case in our baseline life cycle model.

seminal contribution to the study of optimal debt in the standard incomplete market model, on which this paper and others build. Floden (2001) finds that increasing government debt can provide welfare benefits if transfers are below optimal levels. Similarly, Dyrda and Pedroni (2016) find that it is optimal for the government to be a net borrower. However, they find that optimizing both taxes and debt at the same time leads to a smaller level of optimal debt than do previous studies. Relative to these papers, we study optimal public debt and savings. Furthermore, we focus on how optimal policy changes when one considers a life cycle model as opposed to an infinitely lived agent model, and find that including life cycle features has large effects on optimal policy.[2]

Two notable exceptions, Röhrs and Winter (2016) in an infinitely lived agent model and Vogel (2014) in a life cycle model, both find that it can be optimal for the government to be a net saver. In both papers, the government's desire for redistribution partially explains the optimality of public savings, as public savings leads to a lower interest rate and therefore redistributes welfare from wealth-rich agents to wealth-poor agents.[3] This paper finds, as well, that the redistribution motive pushes optimal policy towards public savings across life cycle and infinitely lived agent models. Yet, we find that the redistribution motive is actually weaker in the life cycle model, despite the optimality of public savings in our life cycle model and public debt in our infinitely lived agent model.[4] Instead, the inclusion of an accumulation phase drives the optimality of public savings, as opposed to public debt, in the life cycle model.

This paper is also related to a strand of literature that examines the effects of life cycle features on optimal fiscal policy but generally focuses on taxation instead of government debt. For example, Garriga (2001), Erosa and Gervais

[2]Using infinitely lived agent models, Desbonnet and Weitzenblum (2012), Açikgöz (2015), Dyrda and Pedroni (2016), Röhrs and Winter (2016) find quantitatively large welfare costs of transitioning between steady states after a change in public debt. We do not consider these transitional costs and instead focus on steady state comparisons to more sharply highlight the effect of the life cycle on optimal debt policy.

[3]This motive to redistribute is enhanced in both of these papers since the models are calibrated to match the upper tail of the U.S. wealth distribution, which leads to a small mass of wealth-rich agents and a larger mass of wealth-poor agents.

[4]Specifically, we find that the ratio of lifetime savings income inequality relative to lifetime total income inequality increases with the length of the lifetime (see Dávila et al. (2012) for discussion). Thus, in the infinitely lived agent model, there is a stronger desire for the government to reduce lifetime interest income inequality which they can accomplish through public savings which lowers the interest rate.

(2002) and Conesa et al. (2009) show that introducing a life cycle creates a motive for positive capital taxation, in contrast to the seminal findings of Judd (1985) and Chamley (1986) that optimal capital taxation is zero in the long-run of a class of infinitely lived agent models.[5] With a life cycle, if age-dependent taxation is not feasible then a positive capital may be optimal since it can mimic an age-dependent tax on labor income. Instead of focusing on optimal taxation in a life cycle model, this paper quantifies the effects of life cycle features on optimal government debt.[6] We find that introducing life cycle features changes optimal policy from debt to savings because the accumulation phase reduces to the welfare benefits to government debt, not because the government would like to mimic age-dependent policy.

Finally, our paper is related to Dávila, Hong, Krusell, and Ríos-Rull (2012), whose work defines constrained efficiency in a standard incomplete markets model with infinitely lived agents. Constrained efficient allocations account for the effect of individual behavior on market clearing prices, while satisfying individuals' constraints. The authors show that the price system in the standard incomplete market model does not efficiently allocate resources across agents, and welfare improving equilibrium prices could be attained if agents were to systematically deviate from individually optimal savings and consumption decisions. While this paper does not characterize constrained efficient allocations, this paper's Ramsey allocation improves welfare for similar reasons: since it understands the relationship between public debt and prices, the government can implement a welfare improving allocation that individual agents cannot attain through private markets. As a result of this common mechanism, both of our papers find that a higher capital stock improves welfare. However, Dávila et al. (2012) obtains this result through matching top wealth inequality in an infinitely lived agent model, while our paper does so through adding life cycle features. In the life cycle model, the accumulation phase mitigates the welfare benefits of public debt and leads to the optimality of public savings.

[5]In addition, Aiyagari (1995) and İmrohoroğlu (1998) demonstrate that incomplete markets can overturn the zero capita tax result with uninsurable earnings shocks and sufficiently tight borrowing constraints.

[6]Instead of isolating the effects of life cycle features on optimal debt, Garriga (2001) allows the government to choose sequences for taxes (capital, labor and consumption) as well as government debt. In contrast, our paper explicitly measures how including life cycle features alters optimal debt policy while holding other fiscal instruments constant.

The remainder of this paper is organized as follows. Section 2 illustrates the underlying mechanisms by which optimal government policy interacts with life cycle and infinitely lived agent model features. Section 3 describes the life cycle and infinitely lived agent model environments and defines equilibrium. Section 4 explains the calibration strategy, Section 5 presents quantitative results and Section 6 performs robustness exercises. Section 7 concludes.

2 Illustration of the Mechanisms

In this section, we illustrate the mechanisms that lead the government to an optimal public debt or savings policy. We discuss why optimal government policy may differ in the life cycle and infinitely lived agent models. Specifically, we highlight the distinct savings patterns induced by life cycle features relative to the infinitely lived agent model. Finally, we discuss the main channels by which public debt or savings impacts individual behavior and how the strength of these channels may vary between the two models.

2.1 Life Cycle Phases

In order to highlight how the life cycle may impact optimal debt policy, it will be useful to consider the following illustrative example. Suppose that agents are born with zero wealth, work throughout their lifetimes and die with certainty within a finite number of periods. Agents face idiosyncratic labor productivity shocks and use assets to partially insure against the resulting earnings risk.

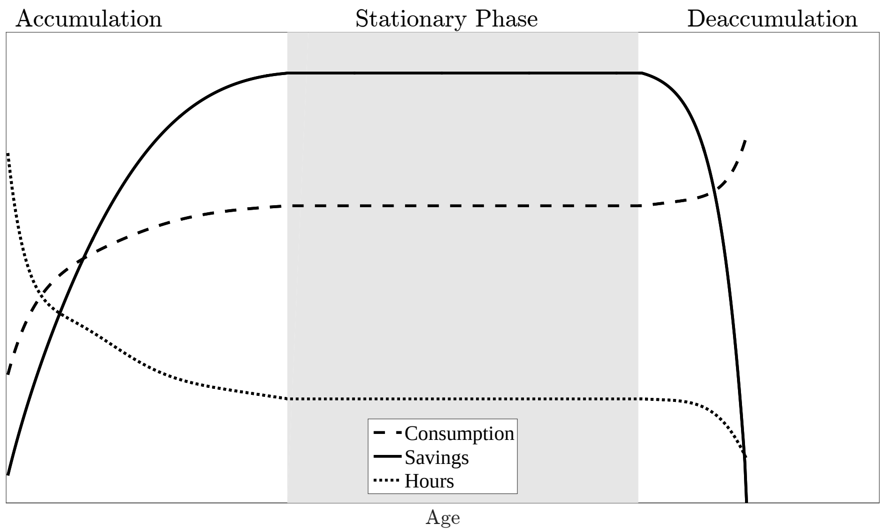
For this hypothetical economy, Figure 1 depicts cross-sectional averages for savings, hours and consumption decisions at each age. Figure 1 shows that agents experience three different phases. Agents enter the economy without any wealth and begin the *accumulation phase*, which is characterized by the accumulation of wealth for precautionary motives.[7] While accumulating a stock of savings, agents tend to work more and consume less.

Once a cohort's average wealth provides sufficient insurance against labor productivity shocks, these agents have entered the *stationary phase*.[8] This phase

[7]Since agents do not retire from supplying labor in this simplified economy, wealth accumulation only provides self-insurance and does not finance post-retirement consumption.

[8]The stationary level of average savings is related to the "target savings level" in Carroll (1992,

Figure 1: Illustrative Example of Life Cycle Phases.



is characterized by savings, hours and consumption that remain constant on average. However, underlying constant averages are agents who respond to shocks by choosing different allocations, thereby moving about various states within a non-degenerate distribution over savings, hours and consumption.

Finally, agents enter the *deaccumulation phase* as they approach the end of their lives. In order to smooth consumption in the final periods of their lives, agents attempt to deaccumulate assets so that they are not forced to consume a large quantity immediately preceding death. Furthermore, with few periods of life remaining, agents no longer want to hold as much savings for precautionary reasons. Thus, the average level of savings and labor supply decreases, while consumption increases slightly.

2.2 Welfare Channels and Life Cycle Features

We identify three main channels through which public debt policy affects welfare: the *direct effect*, the *insurance channel*, and the *inequality channel*. We heuristically characterize how these channels differ across life cycle and infinitely lived

1997). Given the primitives of the economy, an agent faces a tradeoff between consumption levels and consumption smoothing. The agent targets a level of savings that provides sufficient insurance while maximizing expected consumption.

agent economies and lead to different optimal policies.

Direct Effect: The *direct effect* is the partial equilibrium change in the productive capital stock, aggregate consumption and aggregate output with respect to a change in public debt, when holding constant the aggregate labor supply and aggregate private savings. Mechanically, increased public debt crowds out (e.g., decreases) productive capital, thereby generating less output and decreasing aggregate consumption.[9] Generally, decreased aggregate consumption reduces welfare. Absent any general equilibrium effects, this mechanism should operate similarly in both the life cycle and infinitely lived agent economies.

Indirect Effects: While this partial equilibrium channel is a direct effect of policy on aggregate resources, the remaining two channels affect welfare in general equilibrium, that is, by impacting market clearing prices. In particular, decreasing public savings or increasing public debt will crowd out productive capital and lead to an increase in the market clearing interest rate and reduction in the market clearing wage rate.

An increase in the interest rate encourages agents to save. The higher level of savings improves welfare because agents are less likely to face binding liquidity constraints and are, therefore, better insured against labor earnings risk. We refer to this channel as the *insurance channel*.

The insurance channel's welfare benefit varies substantially across the life cycle and infinitely lived agent models. In the infinitely lived agent model, agents exist in a perpetual stationary phase. This implies that when there is a larger steady state aggregate savings stock, agents have a larger amount of savings, on average, at the beginning of each period. Thus, increased public debt improves insurance by providing agents with more *ex ante* precautionary savings. In the life cycle model, in contrast, agents enter the economy with little or no wealth and immediately begin the accumulation phase.[10] While increased public debt may encourage agents to save more over their lifetime, the

[9]By assuming that a representative firm operates a standard Cobb-Douglas production technology, aggregate output is a decreasing function of capital and labor inputs. Standard parameter assumptions ensure that steady state aggregate investment decreases by less than aggregate output decreases upon capital crowd out. Therefore, the resource constraint implies that aggregate consumption decreases.

[10]If life cycle features were introduced in a dynastic model, instead of a life cycle model, where old agents bequeath wealth to agents entering the economy, then the accumulation phase may be more responsive to public policy. Consistent with Fuster, İmrohoroğlu, and İmrohoroğlu

fact that agents need to accumulate this savings mitigates the welfare benefit from debt. A prominent reason for a smaller welfare gain is, for example, that agents would prefer a constant consumption and hours allocation across their lifetimes. However, during the accumulation phase agents work more hours and consume less in order to free resources for more saving. Since the higher interest rate associated with public debt encourages agents to accumulate more savings before reaching the stationary phase, public debt may generate more variation in consumption and hours allocations over the lifetime. Thus, the welfare benefit from the insurance channel tends to be larger in the infinitely lived agent model because it lacks this mitigating effect from the accumulation phase.

The second indirect channel describes the welfare effect of income inequality arising from price changes. Income inequality is composed of both asset and labor income inequality. Since changing public debt has opposite effects on the wage and interest rate, debt policy can be used to reduce total income inequality by lowering inequality from one type of income and raising inequality from the other. The optimal tradeoff depends on the relative contribution to inequality from each source of income. For example, the lower wage associated with public debt will reduce labor income inequality and raise asset income inequality. This tradeoff can reduce lifetime total income inequality when labor income contributes relatively more to income inequality.[11] Similarly, reducing public debt can decrease total income inequality when asset income contributes more to lifetime total income inequality. We refer to this channel as the *inequality channel*.

Whether public debt reduces lifetime total income inequality, or public savings does, varies between the life cycle and infinitely lived agent models. As demonstrated in Dávila, Hong, Krusell, and Ríos-Rull (2012), the contribution of labor income and asset income to lifetime total income inequality depends on agents' lifespan. As agents live longer, lifetime labor income inequality increases because there is a greater chance that agents receive a long string of either positive or negative labor productivity shocks. However, asset income inequality will also develop because agents reduce (increase) their wealth in re-

(2008), the optimal policy differences with the infinitely lived agent model could be smaller since agents would receive some initial wealth through bequests.

[11]While the increase in public debt will increases the interest rate and introduce more inequality from interest income, this effect will be dominated by the lower labor income inequality.

sponse to a string of negative (positive) shocks. Generally, as agents' lifespan increases, asset income becomes a relatively larger contributor to overall income inequality. Accordingly, asset income inequality contributes more to lifetime total income inequality in the infinitely lived agent model, while labor income inequality contributes more to lifetime total income inequality in the life cycle model. Therefore, the inequality channel pushes optimal policy in the life cycle model toward more public debt (less public savings) and pushes optimal policy in the infinitely lived agent model toward more public savings (less public debt).

Overall, higher public debt lowers welfare through the direct channel and raises welfare through the insurance channel, while the inequality channel's effect is ambiguous. The inequality channel moves optimal policy more towards public debt in the life cycle model while moving policy toward more public savings in the infinitely lived agent model. Moreover, the strength of these welfare effects may vary across the life cycle and infinitely lived agent models as a result of differing model features. In particular, the weaker benefit from the insurance channel moves optimal policy towards public savings in the life cycle model.

3 Economic Environment

In this section, we present both the Life Cycle model and the Infinitely Lived Agent model. Given that there are many common features across models, we will first focus on the Life Cycle model in detail before providing an overview of the Infinitely Lived Agent model.

3.1 Life Cycle Model

3.1.1 Production

Assume there exist a large number of firms that sells goods in perfectly competitive product markets, purchase inputs from perfectly competitive factor markets and each operate an identical constant returns to scale production technology, $Y = ZF(K, L)$. These assumptions on primitives admit a representative firm. The representative firm chooses capital (K) and labor (L) inputs in order to maximize profits, given an interest rate r, a wage rate w, a level of total factor productivity Z and capital depreciation rate $\delta \in (0, 1)$.

3.1.2 Consumers

Demographics: Let time be discrete and let each model period represent a year. Each period, the economy is inhabited by J overlapping generations of individuals. We index agents' age in the model by $j = 1, \ldots, J$, where $j = 1$ corresponds to age 21 in the data and J is an exogenously set maximum age (set to age 100 in the data). Before age J all living agents face mortality risk. Conditional on living to age j, agents have a probability s_j of living to age $j+1$, with a terminal age probability given by $s_J = 0$. Each period a new cohort is born and the size of each successive newly born cohort grows at a constant rate $g_n > 0$.

Agents who die before age J may hold savings since mortality is uncertain. These savings are treated as accidental bequests and are equally divided across each living agent in the form of a lump-sum transfer, denoted Tr.

Preferences: Agents rank lifetime paths of consumption and labor, denoted $\{c_j, h_j\}_{j=1}^{J}$, according to the following preferences:

$$\mathbb{E}_1 \sum_{j=1}^{J} \beta^{j-1} s_j \left[u(c_j) - v(h_j, \zeta'_j) \right]$$

where β is the time discount factor. Expectations are taken with respect to the stochastic processes governing labor productivity. Furthermore, $u(c)$ and $v(h)$ are instantaneous utility functions over consumption and labor hours, respectively, satisfying standard conditions. Lastly, ζ'_j is a retirement decision that is described immediately below.

Retirement: Agents choose their retirement age, which is denoted by J_{ret}. A retired agent may not sell labor hours and the decision is irreversible. Agents choose their retirement age in the interval $j \in [\underline{J}_{ret}, \bar{J}_{ret}]$ and are forced to retire after age $\bar{J}_{ret}$. Let $\zeta'_j \equiv \mathbb{1}(j < J_{ret})$ denote an indicator variable that equals one when an agent chooses to continue working and zero upon retirement.

Labor Earnings: Agents are endowed with one unit of time per period, which they split between leisure and market labor. During each period of working life, an agent's labor earnings are we_jh_j, where w is the wage rate per efficiency unit of labor, e_j is the agent's idiosyncratic labor productivity drawn at age j and h_j is the time the agent chooses to work at age j.

Following Kaplan (2012), we assume that labor productivity shocks can be decomposed into four sources:

$$\log(e_j) = \kappa + \theta_j + \nu_j + \epsilon_j$$

where (i) $\kappa \overset{iid}{\sim} \mathcal{N}(0, \sigma_\kappa^2)$ is an individual-specific fixed effect that is drawn at birth, (ii) $\{\theta_j\}_{j=1}^J$ is an age-specific fixed effect, (iii) ν_j is a persistent shock that follows an autoregressive process given by $\nu_{j+1} = \rho \nu_j + \eta_{j+1}$ with $\eta \overset{iid}{\sim} \mathcal{N}(0, \sigma_\nu^2)$ and $\eta_1 = 0$, and (iv) $\epsilon_j \overset{iid}{\sim} \mathcal{N}(0, \sigma_\epsilon^2)$ is a per-period transitory shock.

For notational compactness, we denote the relevant state as a vector $\varepsilon_j = (\kappa, \theta_j, \nu_j, \epsilon_j)$ that contains each element necessary for computing contemporaneous labor earnings, $e_j \equiv e(\varepsilon_j)$, and forming expectations about future labor earnings. Denote the Markov process governing the process for ε by $\pi_j(\varepsilon_{j+1}|\varepsilon_j)$ for each $j = 1, \ldots, \bar{J}_{ret}$ and for each ε_j, ε_{j+1}.

Insurance: Agents have access to a single asset, a non-contingent one-period bond denoted a_j with a market determined rate of return of r. Agents may take on a net debt position, in which case they are subject to a borrowing constraint that requires their debt position be bounded below by $\underline{a} \in \mathbb{R}$. Agents are endowed with zero initial wealth, such that $a_1 = 0$ for each agent.

3.1.3 Government Policy

The government (i) consumes an exogenous amount G, (ii) collects linear Social Security taxes τ_{ss} on all pre-tax labor income below an amount $\bar{x}$, (iii) distributes lump-sum Social Security payments b_{ss} to retired agents, (iv) distributes accidental bequests as lump-sum transfers Tr, and (v) collects income taxes from each individual.

Social Security: The model's Social Security system consists of taxes and payments. The social security payroll tax is given by τ_{ss} with a per-period cap denoted by $\bar{x}$. We assume that half of the social security contributions are paid by the employee and half by the employer. Therefore, the consumer pays a payroll tax given by: $(1/2)\ \tau_{ss} \min\{weh, \bar{x}\}$. Social security payments are computed using an averaged indexed monthly earnings (AIME) that summarizes an agents lifetime labor earnings. Following Huggett and Parra (2010) and Kitao (2014), the AIME is denoted by $\{x_j\}_{j=1}^{J}$ and is given by:

$$x_{j+1} = \left\{ \begin{array}{ll} \frac{1}{j}\left(\min\{we_jh_j, \bar{x}\} + (j-1)x_j\right) & \text{for } j \leq 35 \\ \max\left\{x_j, \frac{1}{j}\left(\min\{we_jh_j, \bar{x}\} + (j-1)x_j\right)\right\} & \text{for } j \in (35, J_{ret}) \\ x_j & \text{for } j \geq J_{ret} \end{array} \right\}$$

The AIME is a state variable for determining future benefits. Benefits consists of a base payment and an adjusted final payment. The base payment, denoted by $b^{ss}_{base}(x_{J_{ret}})$, is computed as a piecewise-linear function over the individual's average labor earnings at retirement $x_{J_{ret}}$:

$$b^{ss}_{base}(x_{J_{ret}}) = \left\{ \begin{array}{lll} \tau_{r1} & \text{for} & x_{J_{ret}} \in [0, b^{ss}_1) \\ \tau_{r2} & \text{for} & x_{J_{ret}} \in [b^{ss}_1, b^{ss}_2) \\ \tau_{r3} & \text{for} & x_{J_{ret}} \in [b^{ss}_2, b^{ss}_3) \end{array} \right\}$$

Lastly, the final payment requires an adjustment that penalizes early retirement and credits delayed retirement. The adjustment is given by:

$$b_{ss}(x_{J_{ret}}) = \left\{ \begin{array}{lll} (1 - D_1(J_{nra} - J_{ret}))b^{ss}_{base}(x_{J_{ret}}) & \text{for} & \underline{J}_{ret} \leq J_{ret} < J_{nra} \\ (1 + D_2(J_{ret} - J_{nra}))b^{ss}_{base}(x_{J_{ret}}) & \text{for} & J_{nra} \leq J_{ret} \leq \bar{J}_{ret} \end{array} \right\}$$

where $D_i(\cdot)$ are functions governing the benefits penalty or credit, $\underline{J}_{ret}$ is the earliest age agents can retire, J_{nra} is the "normal retirement age" and $\bar{J}_{ret}$ is the latest age an agent can retire.

Net Government Transfers: Taxable income is defined as labor income and capital income net of social security contributions from an employer:

$$y(h, a, \varepsilon, \zeta) \equiv \zeta we(\varepsilon)h + r(a + Tr) - \zeta\frac{\tau_{ss}}{2}\min\{we(\varepsilon)h, \bar{x}\}$$

The government taxes each individual's taxable income according to an increasing and concave function, $\Upsilon(y(h, a, e, \zeta))$.

Define the function $T(\cdot)$ as the government's net transfers of income taxes, social security payments and social security payroll taxes to working age agents (if $\zeta = 1$) and retired agents (if $\zeta = 0$). Net transfers are given by:

$$T(h, a, \varepsilon, x, \zeta) = (1 - \zeta) b_{ss}(x) - \zeta \frac{\tau_{ss}}{2} \min\{we(\varepsilon)h, \bar{x}\} - \Upsilon(y(h, a, \varepsilon, \zeta))$$

Public Savings and Budget Balance: Each period, the government accumulates savings, denoted B', and collects asset income rB. The resulting government budget constraint is:

$$G + B' - B = rB + \Upsilon_y \tag{1}$$

where Υ_y is aggregate revenues from income taxation and G is an unproductive level of government expenditures.[12] The model's Social Security system is self-financing and therefore does not appear in the governmental budget constraint.

3.1.4 Consumer's Problem

The agent's state variables consist of asset holdings a, labor productivity shocks $\varepsilon \equiv (\kappa, \theta, \nu, \epsilon)$, Social Security contribution (AIME) variable x and retirement status ζ. For age $j \in \{1, \ldots, J\}$, the agent's recursive problem is:

$$V_j(a, \varepsilon, x, \zeta) = \max_{c, a', h, \zeta'} \left[u(c) - v(h, \zeta') \right] + \beta s_j \sum_{\varepsilon'} \pi_j(\varepsilon' | \varepsilon) V_{j+1}(a', \varepsilon', x', \zeta') \tag{2}$$

$$\begin{aligned} \text{s.t.} \quad c + a' &\leq \zeta' we(\varepsilon)h + (1 + r)(a + Tr) + T(h, a, \varepsilon, x, \zeta') \\ a' &\geq \underline{a} \\ \zeta' &\in \{\mathbb{1}(j < \underline{J}_{ret}),\ \mathbb{1}(j \leq \bar{J}_{ret}) \cdot \zeta\} \end{aligned}$$

The indicator function $\mathbb{1}(j < \underline{J}_{ret})$ equals one when an agent is too young to retire and equals zero thereafter. Additionally $\mathbb{1}(j \leq \bar{J}_{ret})$ equals zero for all ages after an agent must retire and equals one beforehand. Therefore the agent's recursive problem nests all three stages of life: working life, near-retirement and

[12] Two recent papers, Röhrs and Winter (2016) and Chaterjee, Gibson, and Rioja (2016) have relaxed the standard Ramsey assumption that government expenditures are unproductive. Both papers show that public savings is optimal with productive government expenditures, intuitively because there is an additional benefit to aggregate output.

retirement.[13]

3.1.5 Recursive Competitive Equilibrium

Agents are heterogeneous with respect to their age $j \in \mathbf{J} \equiv \{1, \ldots, J\}$, wealth $a \in \mathbf{A}$, labor productivity $\varepsilon \in \mathbf{E}$, average lifetime earnings $x \in \mathbf{X}$ and retirement status $\zeta \in \mathbf{R} \equiv \{0, 1\}$. Let $\mathbf{S} \equiv \mathbf{A} \times \mathbf{E} \times \mathbf{X} \times \mathbf{R}$ be the state space and $\mathcal{B}(\mathbf{S})$ be the Borel σ-algebra on $\mathbf{S}$. Let $\mathbf{M}$ be the set of probability measures on $(\mathbf{S}, \mathcal{B}(\mathbf{S}))$. Then $(\mathbf{S}, \mathcal{B}(\mathbf{S}), \lambda_j)$ is a probability space in which $\lambda_j(S) \in \mathbf{M}$ is a probability measure defined on subsets of the state space, $S \in \mathcal{B}(\mathbf{S})$, that describes the distribution of individual states across age-j agents. Denote the fraction of the population that is age $j \in \mathbf{J}$ by μ_j. For each set $S \in \mathcal{B}(\mathbf{S})$, $\mu_j \lambda_j(S)$ is the fraction of age $j \in \mathbf{J}$ and type $S \in \mathbf{S}$ agents in the economy. We can now define a recursive competitive equilibrium of the economy.

Definition (Equilibrium): Given a government policy $(G, B, B', \Upsilon, \tau_{ss}, b_{ss})$, a *stationary recursive competitive equilibrium* is (i) an allocation for consumers described by policy functions $\{c_j, a'_j, h_j, \zeta'_j\}_{j=1}^J$ and consumer value function $\{V_j\}_{j=1}^J$, (ii) an allocation for the representative firm (K, L), (iii) prices (w, r), (iv) accidental bequests Tr, and (v) distributions over agents' state vector at each age $\{\lambda_j\}_{j=1}^J$ that satisfy:

(a) Given prices, policies and accidental bequests, $V_j(a, \varepsilon, x)$ solves the Bellman equation (2) with associated policy functions $c_j(a, \varepsilon, x, \zeta)$, $a'_j(a, \varepsilon, x, \zeta)$, $h_j(a, \varepsilon, x, \zeta)$ and $\zeta'_j(a, \varepsilon, x, \zeta)$.

(b) Given prices (w, r), the representative firm's allocation minimizes cost: $r = ZF_K(K, L) - \delta$ and $w = ZF_L(K, L)$

(c) Accidental bequests, Tr, from agents who die at the end of this period are distributed equally across next period's living agents:

$$(1 + g_n)Tr = \sum_{j=1}^{J} (1 - s_j)\mu_j \int a'_j(a, \varepsilon, x, \zeta) d\lambda_j(a, \varepsilon, x, \zeta)$$

[13] During an agent's working life (ages $j < \underline{J}_{ret}$) the agent's choice set for retirement is $\zeta' \in \{1, 1\}$ and therefore the agent must continue working. Near retirement (ages $\underline{J}_{ret} \le j \le \bar{J}_{ret}$), the agent's choice set is $\zeta' \in \{0, 1\}$ and the agent may retire by choosing $\zeta' = 0$. Lastly, if an agent has retired either because he chose retirement at a previous date ($\zeta = 0$) or because of mandatory retirement ($j > \bar{J}_{ret}$), then the choice set is $\{0, 0\}$ and $\zeta' = \zeta = 0$.

(d) Government policies satisfy budget balance in equation (1), where aggregate income tax revenue is given by:

$$\Upsilon_y \equiv \sum_{j=1}^{J} \mu_j \int \Upsilon \left(y\big(h_j(a,\varepsilon,x,\zeta), a, \varepsilon, \zeta_j'(a,\varepsilon,x,\zeta)\big) \right) \, d\lambda_j(a,\varepsilon,x,\zeta)$$

(e) Social security is self-financing:

$$\sum_{j=1}^{J} \mu_j \int \zeta_j'(a,\varepsilon,x,\zeta)\tau_{ss} \min\{we(\varepsilon)h_j(a,\varepsilon,x,\zeta), \bar{x}\} d\lambda_j(a,\varepsilon,x,\zeta)$$

$$= \sum_{j=1}^{J} \mu_j \int (1 - \zeta_j'(a,\varepsilon,x,\zeta)) b_{ss}(x) d\lambda_j(a,\varepsilon,x,\zeta) \tag{3}$$

(f) Given policies and allocations, prices clear asset and labor markets:

$$K - B = \sum_{j=1}^{J} \mu_j \int a \, d\lambda_j(a,\varepsilon,x,\zeta)$$

$$L = \sum_{j=1}^{J} \mu_j \int \zeta_j'(a,\varepsilon,x,\zeta) e(\varepsilon) h_j(a,\varepsilon,x,\zeta) \, d\lambda_j(a,\varepsilon,x,\zeta)$$

and the allocation satisfies the resource constraint (guaranteed by Walras' Law):

$$\sum_{j=1}^{J} \mu_j \int c_j(a,\varepsilon,x,\zeta) d\lambda_j(a,\varepsilon,x,\zeta) + G + K' = ZF(K,L) + (1-\delta)K$$

(g) Given consumer policy functions, distributions across age j agents $\{\lambda_j\}_{j=1}^{J}$ are given recursively from the law of motion $T_j^* : \mathbf{M} \to \mathbf{M}$ for all $j \in \mathbf{J}$ such that T_j^* is given by:

$$\lambda_{j+1}(\mathcal{A} \times \mathcal{E} \times \mathcal{X} \times \mathcal{R}) = \sum_{\zeta \in \{0,1\}} \int_{A \times E \times X} Q_j\left((a,\varepsilon,x,\zeta), \mathcal{A} \times \mathcal{E} \times \mathcal{X} \times \mathcal{R}\right) d\lambda_j$$

where $\mathcal{S} \equiv \mathcal{A} \times \mathcal{E} \times \mathcal{X} \times \mathcal{R} \subset \mathbf{S}$, and $Q_j : \mathbf{S} \times \mathcal{B}(\mathbf{S}) \to [0,1]$ is a transition function on $(\mathbf{S}, \mathcal{B}(\mathbf{S}))$ that gives the probability that an age-j agent with current state $\mathbf{s} \equiv (a, \varepsilon, x, \zeta)$ transits to the set $\mathcal{S} \subset \mathbf{S}$ at age $j+1$. The transition function is given by:

$$Q_j\left((a, \varepsilon, x, \zeta), \mathcal{S}\right) = \left\{ \begin{array}{cl} s_j \cdot \pi_j(\mathcal{E}|\varepsilon)^{\zeta} & \text{if } a'_j(\mathbf{s}) \in \mathcal{A},\ x'_j(\mathbf{s}) \in \mathcal{X},\ \zeta'_j(\mathbf{s}) \in \mathcal{R} \\ 0 & \text{otherwise} \end{array} \right\}$$

where agents that continue working and transition to set $\mathcal{E}$ choose $\zeta'_j(\mathbf{s}) = 1$, while agents that transition from working life to retirement choose $\zeta'_j(\mathbf{s}) = 0$. For $j = 1$, the distribution λ_j reflects the invariant distribution $\pi_{ss}(\varepsilon)$ of initial labor productivity over $\varepsilon = (\kappa, \theta_1, 0, \epsilon_1)$.

(h) Aggregate capital, governmental debt, prices and the distribution over consumers are stationary, such that $K' = K$, $B' = B$, $w' = w$, $r' = r$, and $\lambda'_j = \lambda_j$ for all $j \in \mathbf{J}$.

3.2 Infinitely Lived Agent Model

The infinitely lived agent model differs from the life cycle model in three ways. First, agents in the infinitely lived agent model have no mortality risk ($s_j = 1$ for all $j \geq 1$) and lifetimes are infinite ($J \to \infty$). Second, labor productivity no longer has an age-dependent component ($\theta_j = \bar{\theta}$ for all $j \geq 1$). Lastly, there is no retirement ($\underline{J}_{ret} \to \infty$ such that $\zeta_j = 1$ for all $j \geq 1$) and there is no Social Security program ($\tau_{ss} = 0$ and $b_{ss}(x) = 0$ for all x).

Accordingly, we study a stationary recursive competitive equilibrium in which the initial endowment of wealth and labor productivity shocks no longer affects individual decisions and the distribution over wealth and labor productivity is time invariant.

Definition (Equilibrium): Given a government policy (G, B, B', Υ), a *stationary recursive competitive equilibrium* is (i) an allocation for consumers described by policy functions (c, a', h) and consumer value function V, (ii) an allocation for the representative firm (K, L), (iii) prices (w, r), and (v) a distribution over agents' state vector λ that satisfy:

(a) Given prices and policies, $V(a,\varepsilon)$ solves the following Bellman equation:

$$V(a,\varepsilon) = \max_{c,a',h} \left[u(c) - v(h)\right] + \beta \sum_{\varepsilon'} \pi(\varepsilon'|\varepsilon) V(a', \varepsilon') \tag{4}$$

$$\begin{aligned} \text{s.t.} \quad c + a' &\leq we(\varepsilon)h + (1+r)a + \Upsilon(y(h,a,\varepsilon)) \\ a' &\geq \underline{a} \end{aligned}$$

with associated policy functions $c(a,\varepsilon)$, $a'(a,\varepsilon)$ and $h(a,\varepsilon)$.

(b) Given prices (w,r), the representative firm's allocation minimizes cost.

(c) Government policies satisfy budget balance in equation (1), where aggregate income tax revenue is given by:

$$\Upsilon_y \equiv \int \Upsilon\big(y(h(a,\varepsilon),a,\varepsilon)\big)\, d\lambda(a,\varepsilon)$$

(d) Given policies and allocations, prices clear asset and labor markets:

$$K - B = \int a \, d\lambda(a,\varepsilon)$$

$$L = \int e(\varepsilon)h(a,\varepsilon)\, d\lambda(a,\varepsilon)$$

and the allocation satisfies the resource constraint (guaranteed by Walras' Law):

$$\int c(a,\varepsilon) d\lambda(a,\varepsilon) + G + K' = ZF(K,L) + (1-\delta)K$$

(e) Given consumer policy functions, the distribution over wealth and productivity shocks is given recursively from the law of motion $T^* : \mathbf{M} \to \mathbf{M}$ such that T^* is given by:

$$\lambda'(\mathcal{A} \times \mathcal{E}) = \int_{A \times E} Q_j\left((a,\varepsilon), \mathcal{A} \times \mathcal{E}\right) d\lambda$$

where $\mathcal{S} \equiv \mathcal{A} \times \mathcal{E} \subset \mathbf{S}$, and $Q : \mathbf{S} \times \mathcal{B}(\mathbf{S}) \to [0,1]$ is a transition function on $(\mathbf{S}, \mathcal{B}(\mathbf{S}))$ that gives the probability that an agent with current state $\mathbf{s} \equiv (a,\varepsilon)$ transits to the set $\mathcal{S} \subset \mathbf{S}$ in the next period. The transition function is given by:

$$Q\left((a,\varepsilon),\mathcal{S}\right)=\left\{\begin{array}{cc}\pi(\mathcal{E}|\varepsilon) & \text{if } a'(\mathbf{s})\in\mathcal{A}, \\ 0 & \text{otherwise}\end{array}\right\}$$

(f) Aggregate capital, governmental debt, prices and the distribution over consumers are stationary, such that $K' = K$, $B' = B$, $w' = w$, $r' = r$, and $\lambda' = \lambda$.

3.3 Balanced Growth Path

Following Aiyagari and McGrattan (1998), we will further assume that total factor productivity, Z, grows over time at rate $g_z > 0$. In both the life cycle model and infinitely lived agent model, we will study a balanced growth path equilibrium in which all aggregate variables grow at the same rate as output. Denote the growth rate of output as g_y. Refer to Appendix A.1 for a formal construction of the balanced growth path for this set of economies.

4 Calibration

In this section we calibrate the life cycle model and then discuss the parameter values that are different in the infinitely lived agent model. Overall, one subset of parameters are assigned values without needing to solve the model. These parameters are generally the same in both models. The other subset of parameters are estimated using a simulated method of moments procedure that minimizes the distance between model generated moments and empirical ones. We allow these parameters to vary across the models while matching the same moments in the two models. Table 1 summarizes the target and value for each parameter.

Demographics: Agents enter the economy at age 21 (or model age $j = 1$) and exogenously die at age 100 (or model age $J = 81$). We set the conditional survival probabilities $\{s_j\}_{j=1}^{J}$ according to Bell and Miller (2002) and impose $s_J = 0$. We set the population growth rate to $g_n = 0.011$ to match annual population growth in the US.

Production: Given that $Y = ZF(K, L)$, the production function is assumed to be Cobb-Douglas of the form $F(K, L) = K^{\alpha}L^{1-\alpha}$ where $\alpha = 0.36$ is the income share accruing to capital. The depreciation rate is to $\delta = 0.0833$ which allows the model to match the empirically observed investment-to-output ratio.

Preferences: The utility function is separable in the utility over consumption and disutility over labor (including retirement):

$$u(c) - v(h, \zeta') = \frac{c^{1-\sigma}}{1-\sigma} - \left(\chi_1 \frac{h^{1+\frac{1}{\gamma}}}{1+\frac{1}{\gamma}} + \zeta' \chi_2 \right) .$$

Utility over consumption is a CRRA specification with a coefficient of relative risk aversion $\sigma = 2$, which is consistent with Conesa et al. (2009) and Aiyagari and McGrattan (1998). Disutility over labor exhibits a constant intensive margin Frisch elasticity. We choose $\gamma = 0.5$ such that the Frisch elasticity consistent with the majority of the related literature as well as the estimates in Kaplan (2012).

We calibrate the labor disutility parameter χ_1 so that the cross sectional average of hours is one third of the time endowment. Finally, χ_2 is a fixed utility cost of earning labor income before retirement. The fixed cost generates an extensive margin decision through a non-convexity in the utility function. We choose χ_2 to match the empirical observation that seventy percent of the population has retired by the normal retirement age.

Labor Productivity Process: We take the labor productivity process from the estimates in Kaplan (2012) based on the estimates from the PSID data.[14] The deterministic labor productivity profile, $\{\theta_j\}_{j=1}^{\bar{J}_{ret}}$, is (i) smoothed by fitting a quadratic function in age, (ii) normalized such that the value equals unity when an agent enters the economy, and (iii) extended to cover ages 21 through 70 which we define as the last period in which agents are assumed to be able to participate in the labor activities ($\bar{J}_{ret}$).[15] The permanent, persistent, and transitory idiosyncratic

[14]For details on estimation of this process, see Appendix E in Kaplan (2012). A well known problem with a log-normal income process is that the model generated wealth inequality does not match that in the data, primarily due to missing the fat upper tail of the distribution. However, Röhrs and Winter (2016) demonstrate that when the income process in an infinitely lived agent model is altered to match the both the labor earnings and wealth distributions (quintiles and gini coefficients), the change in optimal policy is relatively small, with approximately 30 percentage points due to changing the income process and the remaining 110 percentage points due to changing borrowing limits, taxes and invariant parameters (such as risk aversion, the Frisch elasticity, output growth rate and depreciation).

[15]The estimates in Kaplan (2012) are available for ages 25-65.

shocks to individual's productivity are normally distributed with zero mean. The remaining parameters are also set in accordance with the Kaplan's (2012) estimates: $\rho = 0.958$, $\sigma_\kappa^2 = 0.065$, $\sigma_\nu^2 = 0.017$ and $\sigma_\epsilon^2 = 0.081$.

Government: Consistent with Aiyagari and McGrattan (1998) we set government debt equal to two-thirds of output. We set government consumption equal to 15.5 percent of output consistent. This ratio corresponds to the average of government expenditures to GDP from 1998 through 2007.[16]

Income Taxation: The income tax function and parameter values are from Gouveia and Strauss (1994). The functional form is:

$$\Upsilon(y) = \tau_0 \left(y - \left(y^{-\tau_1} + \tau_2 \right)^{-\frac{1}{\tau_1}} \right)$$

The authors find that $\tau_0 = 0.258$ and $\tau_1 = 0.768$ closely match the U.S. tax data. When calibrating the model we set τ_2 such that the government budget constraint is satisfied.

Social Security: We set the normal retirement age to 66. Consistent with the minimum and maximum retirement ages in the U.S. Social Security system, we set the interval in which agents can retire to the ages 62 and 70. The early retirement penalty and later retirement credits are set in accordance with the Social Security program. In particular, if agents retire up to three years before the normal retirement age agents benefits are reduced by 6.7 percent for each year they retire early. If they choose to retire four or five years before the normal retirement age benefits are reduced by an additional 5 percent for these years. If agents choose to delay retirement past normal retirement age then their benefits are increased by 8 percent for each year they delay. The marginal replacement rates in the progressive Social Security payment schedule ($\tau_{r1}, \tau_{r2}, \tau_{r3}$) are also set at their actual respective values of 0.9, 0.32 and 0.15. The bend points where the marginal replacement rates change (b_1^{ss}, b_2^{ss}, b_3^{ss}) and the maximum earnings ($\bar{x}$) are set equal to the actual multiples of mean earnings used in the U.S. Social Security system so that b_1^{ss}, b_2^{ss} and $b_3^{ss} = \bar{x}$ occur at 0.21, 1.29 and 2.42 times average earnings in the economy. We set the payroll tax rate, τ_{ss} such that the

[16]We exclude government expenditures on Social Security since they are explicitly included in our model.

Table 1: Calibration Targets and Parameters for Baseline Economy.

Description	Parameter	Value	Target or Source
Demographics			
Maximum Age	J	81 (100)	By Assumption
Min/Max Retirement Age	$\underline{J}_{ret}, \bar{J}_{ret}$	43, 51 (62, 70)	Social Security Program
Population Growth	g_n	1.1%	Conesa et al (2009)
Survival Rate	$\{s_j\}_{j=1}^{J}$	—	Bell and Miller (2002)
Preferences and Borrowing			
Coefficient of RRA	σ	2.0	Kaplan (2012)
Frisch Elasticity	γ	0.5	Kaplan (2012)
Coefficient of Labor Disutility	χ_1	55.3	Avg. Hours Worked = 1/3
Fixed Utility Cost of Labor	χ_2	1.038	70% retire by NRA
Discount Factor	β	1.012	Capital/Output = 2.7
Borrowing Limit	$\underline{a}$	0	By Assumption
Technology			
Capital Share	α	0.36	NIPA
Capital Depreciation Rate	δ	0.0833	Investment/Output = 0.255
Productivity Level	Z	1	Normalization
Output Growth	g_y	1.85%	NIPA
Labor Productivity			
Persistent Shock, autocorrelation	ρ	0.958	Kaplan (2012)
Persistent Shock, variance	σ_ν^2	0.017	Kaplan (2012)
Permanent Shock, variance	σ_κ^2	0.065	Kaplan (2012)
Transitory Shock, variance	σ_ϵ^2	0.081	Kaplan (2012)
Mean Earnings, Age Profile	$\{\theta\}_{j=1}^{\bar{J}_{ret}}$	—	Kaplan (2012)
Government Budget			
Government Consumption	G/Y	0.155	NIPA Average 1998-2007
Government Savings	B/Y	-0.667	NIPA Average 1998-2007
Marginal Income Tax	τ_0	0.258	Gouveia and Strauss (1994)
Income Tax Progressivity	τ_1	0.786	Gouveia and Strauss (1994)
Income Tax Progressivity	τ_2	4.541	Balanced Budget
Social Security			
Payroll Tax	τ_{ss}	0.103	Social Security Program
SS Replacement Rates	$\{\tau_{ri}\}_{i=1}^{3}$	See Text	Social Security Program
SS Replacement Bend Points	$\{b_i^{ss}\}_{i=1}^{3}$	See Text	Social Security Program
SS Early Retirement Penalty	$\{\kappa_i\}_{i=1}^{3}$	See Text	Social Security Program

program's budget is balanced. In our baseline model the payroll tax rate is 10.3 percent, roughly equivalent with the statutory rate.[17]

Infinitely Lived Agent Model: The infinitely lived agent model does not have an age-dependent wage profile. For comparability across models, we replace the

[17] Although the payroll tax rate in the U.S. economy is slightly higher than our calibrated value, the OASDI program includes additional features outside of the retirement benefits.

age-dependent wage profile with the population-weighted average of θ_j's, such that $\bar{\theta} = \sum_{j=1}^{\bar{J}_{ret}} (\mu_j / \sum_{j=1}^{\bar{J}_{ret}} \mu_j)\theta_j \approx 1.86$.[18] In the absence of a retirement decision, we set $\chi_2 = 0$. Lastly, we recalibrate the parameters (β, χ_1) to the same targets as in the life cycle model and choose τ_2 to balance the government's budget.

5 Quantitative Effects of the Life Cycle on Optimal Policy

Having described how we use external data to discipline the models' structural parameters, we use the calibrated model to measure optimal policy across the life cycle and infinitely lived agent models. Then we perform a series of counterfactual experiments to highlight the mechanisms that generate differences in optimal policy across the models.

5.1 Optimal Public Policy

The government maximizes social welfare by choosing a budget feasible level of public savings, B, subject to allocations being a stationary recursive competitive equilibrium. We consider an ex-ante Utilitarian social welfare criterion that evaluates the expected lifetime utility of an agent that has yet to enter the steady state economy.[19] For the life cycle model, the government's welfare maximization problem is:

$$S_J(V_1, \lambda_1) \equiv \max_B \left\{ \int V_1(a, \varepsilon, x, \zeta; B)\; d\lambda_1(a, \varepsilon, x, \zeta; B) \quad \text{s.t.} \quad (1),\ (3) \right\}$$

[18]When calibrating the stochastic process for idiosyncratic productivity shocks, we use the same process in the both the life cycle and infinitely lived agent models. Using the same underlying process will imply that cross-sectional wealth inequality will be different across the two models. One reason is that the life cycle model will have additional cross-sectional inequality due to the humped shaped savings profiles, which induces the accumulation, stationary, and deaccumulation phases. We view these difference in inequality as a fundamental difference between the two models and, therefore, choose not to specially alter the infinitely lived agent model to match a higher level of cross-sectional inequality.

[19]Our analysis focuses on welfare across steady states. This analysis omits the transitional costs between steady states which can be large. See Domeij and Heathcote (2004), Fehr and Kindermann (2015) and Dyrda and Pedroni (2016).

where the value function $V_1(\cdot;B)$, distribution function $\lambda_1(\cdot;B)$ and policy functions embedded in equations (1) and (3) are determined in competitive equilibrium and depend on the government's choice of public savings. Furthermore, $B' = B$ in steady state. Since the distribution of taxable income and tax revenues depend on public savings, we adjust the Social Security payroll tax rate τ_{ss} to ensure that Social Security is self-financing and, furthermore, adjust the income tax parameter τ_0 to ensure that the government budget is balanced.[20]

For the infinitely lived agent model, the government's welfare maximization problem is:

$$S_\infty(V,\lambda) \equiv \max_B \left\{ \int V(a,\varepsilon;B)\, d\lambda(a,\varepsilon;B) \quad \text{s.t.} \quad G = rB + \Upsilon_y(\tau_0, B) \right\}$$

The infinitely lived agent model government's welfare maximization problem is nearly identical to that of the life cycle model's, except that the value function and distribution function do not depend on age and there is no Social Security program, so that equation (3) does not define the feasible set.

We find that the two models generate starkly different optimal policies, which are reported in Table 2. In the infinitely lived agent model, the government optimally holds debt equal to 24 percent of output.[21] In the life cycle model, on the other hand, the government optimally holds savings equal to 59 percent of output. Thus, including life cycle features causes optimal policy to switch from public debt to savings, with an 80 percentage point swing in optimal policy.

5.2 The Effect of Life Cycle Features on Optimal Policy

There are three main differences between the life cycle and infinitely lived agent models: (i) agents in the life cycle model experience an accumulation phase while agents in the infinitely lived agent model experience a perpetual stationary

[20] We choose to use τ_0 to balance the government budget instead of the other income taxation parameters (τ_1, τ_2) so that the average income tax rate is used to clear the budget, as opposed to changing in the progressivity of the income tax policy. The average tax rate is the closest analogue to the flat tax that Aiyagari and McGrattan (1998) use to balance the government's budget in their model.

[21] This is generally consistent with Aiyagari and McGrattan's (1998) optimal policy. This paper assumes a different stochastic process governing labor productivity, a different utility function, non-linear income taxation and different parameter values. A quantitative decomposition of these model differences are available upon request.

Table 2: Aggregates and Prices Across Models

	Life Cycle		Infinitely Lived	
	Baseline	**Optimal**	**Baseline**	**Optimal**
Public Savings/Output	**-0.67**	**0.59**	**-0.67**	**-0.24**
Consumption	0.53	0.54	0.66	0.66
Output	0.93	1.01	1.16	1.18
Labor	0.53	0.54	0.66	0.67
Productive Capital	2.5	3.03	3.13	3.23
Private Savings	3.12	2.43	3.9	3.51
Public Savings	-0.62	0.6	-0.77	-0.28
Interest Rate	5.0%	3.6%	5.0%	4.8%
Wage	1.12	1.19	1.12	1.13

phase, (ii) age-dependent features in the life cycle model, such as mortality risk, an age-dependent wage profile, retirement and Social Security, do not exist in the infinitely lived agent model and (iii) the lifespan is different in the two models. We begin by demonstrating that the introduction of the accumulation phase in the life cycle model can more than explain the differences in optimal policies between the two models. Then we systematically decompose the effects of each of the three model differences on optimal policy.

5.2.1 The Accumulation Phase

This section quantifies the importance of the accumulation phase for explaining the difference in optimal policies between the life cycle and infinitely lived agent models. We do this by constructing an approximation to the infinitely lived agent economy that features an accumulation phase. Relative to the infinitely lived agent model, the counterfactual model mainly differs from the infinitely lived agent model in that agents are endowed with zero wealth. In order to activate the accumulation phase we assume agents have finite lifespans. However, we assume that agents die at the end of $J = 1000$ periods, a sufficiently large terminal age to mimic the infinitely lived agent model. Therefore, by construction, the fundamental difference between the counterfactual model and the infinitely lived agent model is the accumulation phase.[22]

Using the calibrated counterfactual model, we conduct a computational experiment to isolate the impact of the accumulation phase on optimal policy. Suppose that the government chooses policy according to an alternative social welfare criterion that places less weight on the flow of utility during youth than does the ex ante Utilitarian welfare criterion. In particular, suppose that the alternative social welfare criterion only incorporates the expected present value of utility after a given age $j^* > 1$, and ignores the flow of utility from ages 1 to $j^* - 1$. Government policy, therefore, maximizes agents' expected utility as of age j^*, subject to allocations being determined in competitive equilibrium:

$$\tilde{S}(V_{j^*}, \lambda_{j^*}) \equiv \max_{B} \left\{ \int V_{j^*}(a, \varepsilon; B)\, d\lambda_{j^*}(a, \varepsilon; B) \quad \text{s.t.} \quad G = rB + \Upsilon_y(\tau_0, B) \right\}.$$

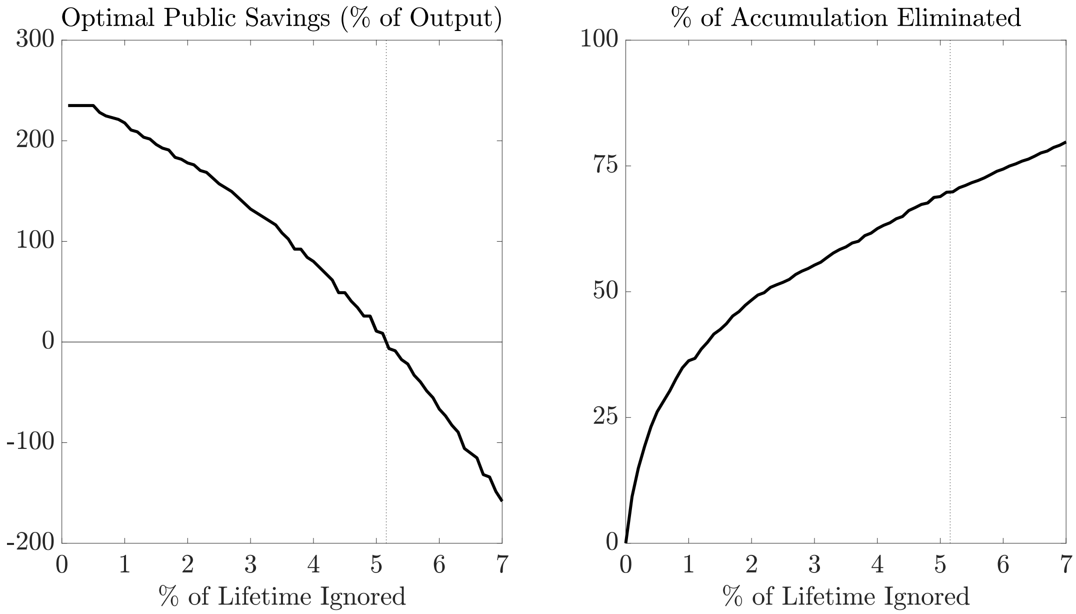
Figure 2 plots the optimal policy under this alternative welfare criterion as a function of the percent of the lifetime that threshold age, j^*, represents. We observe that optimal policy monotonically decreases from the public savings to output ratio of 2.35 when all of the lifetime is considered, to an optimal debt policy when the social welfare function ignores at least 5.2 percent of agents' early lifetime. Across models, higher public debt (lower public savings) crowds out the productive capital stock and leads to a higher interest rate. The higher interest rate encourages agents to save, which improves self insurance. When the age threshold is small, the government includes agents' utility during the accumulation phase in its welfare maximization calculation. As a result, agents' welfare improvement from accumulating more precautionary savings is offset by the utility cost of accumulating that savings. However, when the age threshold is large, the government ignores the flow of utility for agents in the accumulation phase and there is only a large welfare improvement from agents living with more self insurance. The right panel of Figure 2 shows that ignoring at least 5.2 percent of agents' early lifetime corresponds to ignoring at least 69 percent of the accumulation phase.

The experiment shows that the existence of an accumulation phase is crucial to the optimality of public savings. Without the accumulation phase, the benefits

[22] Neither the infinitely lived nor the counterfactual model feature any age-dependent features (e.g., no mortality risk, no age-dependent wage profile, no retirement and no Social Security). In order to make quantitative comparisons across models, the counterfactual model's parameters are recalibrated to match all relevant the targets described in Section 4.

of the insurance channel strengthen and lead to the optimality of public debt. In contrast, when the accumulation phase is incorporated, the benefits from the insurance channel are smaller and public savings is optimal.

Figure 2: Optimal Policy and Eliminating Accumulation



Notes: The left panel graphs the optimal public savings to output ratio (y-axis) associated with ignoring a given percent of early life utility flows (x-axis). The percent of "Lifetime Ignored" is measured as $100 \cdot (j^*/J)$, using the given value of j^* and $J = 1000$. The right panel graphs the percent of accumulation that is eliminated under the optimal policy associated with ignoring a given percent of early life utility flows. The percent of eliminated wealth accumulation is defined as the average private savings of j^*-age agents relative to the peak average savings and converted to a percent, given a particular optimal public savings policy. The vertical dashed line demarcates the percent of early lifetime utility ignored at which optimal policy switches from public savings to debt.

5.2.2 Decomposing the Effects of Life Cycle Features

Apart from the accumulation phase, there are two remaining differences between the life cycle and infinitely lived agent models: lifespan and age-dependent features (e.g., mortality risk, age-dependent wage profile, retirement and Social Security). Next, we quantify the effect of each of these differences on optimal policy. Unlike removing the accumulation phase, we find that removing lifespan and age-dependent features shifts optimal policy towards *more* public savings.

Table 3: Optimal Public Savings-to-Output Ratios

Life Cycle	Counterfactual Models: No Age Features (81 periods)	Counterfactual Models: Long Lifespan (401 periods)	Infinitely Lived
0.59	2.00	2.48	-0.24

In order to characterize the individual effects of these differences on optimal policy between the life cycle and infinitely lived agent models, we construct two counterfactual economies. The first is the "No Age-Dependent Features" economy, which is a version of the life cycle model that excludes all age-dependent features (e.g., no mortality risk, no age-dependent wage profile, no retirement and no Social Security), while maintaining the lifespan of $J = 81$ periods. The second is the "Long Lifespan" economy, which removes age-dependent model features and also extends agents' lifetime to $J = 401$ periods.[23]

Table 3 reports optimal policies in the benchmark life cycle, infinitely lived, and counterfactual models. First, comparing the baseline life cycle model and "No Age-Dependent Features" economy isolates the effect of age-dependent features, which leads to an increase in the working lifetime due to the removal of retirement and mortality. We find that the optimal savings-to-output shifts from 59 to 200 percent. Comparing the "No Age-Dependent Features" and "Long Lifespan" counterfactual economies isolates the effect of further increasing agents' working lifetime and lifespan. This effect additionally increases the optimal savings-to-output from 200 to 248 percent. Finally, comparing the "Long Lifespan" economy with the infinitely lived agent model highlights the effect of the accumulation phase on optimal policy, which switches optimal policy from savings to debt-to-output of 24 percent.

Removing age-dependent features and extending agents' lifespan generates increases in optimal public savings by lengthening the span of life that agents spend working. In the life cycle model, there is a tendency for wealth inequality to increase with an extension of agents' expected working lifespan and, in turn, this generates a greater amount of inequality in lifetime asset income. Table 4(a) reports that, indeed, measures of lifetime asset income inequality (the Gini co-

[23]In order to make quantitative comparisons across models, each counterfactual model's parameters are recalibrated to match all relevant the targets described in Section 4.

Table 4: Lifetime Income Inequality

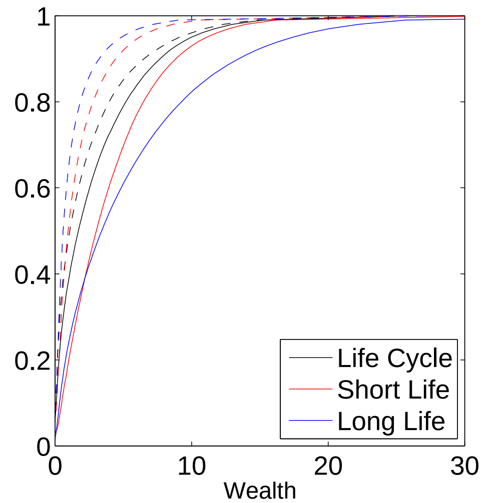
		Counterfactuals	
	Life Cycle	**No Age Features (81 periods)**	**Long Lifespan (401 periods)**
(a) Asset Income Inequality in Baseline Calibration			
Coefficient of Variation	0.33	0.34	0.49
Gini Coefficient	0.19	0.20	0.28
(b) Lifetime Total Income Inequality			
Coefficient of Variation			
Baseline	0.36	0.32	0.31
Optimal	0.35	0.30	0.27
% Change	-1.8%	-7.1%	-13.8%
Gini Coefficient			
Baseline	0.20	0.18	0.17
Optimal	0.19	0.16	0.15
% Change	-2.0%	-6.9%	-12.0%

efficient and the coefficient of variation), increase under the baseline calibration when these two features are removed.[24] Likewise, under the baseline policy, the cumulative distribution functions for wealth across all agents (see Figure 3) demonstrate that wealth becomes more unequal when either of these features are removed. In contrast, Table 4(b) demonstrates that lifetime *total* income inequality tends to decrease. Therefore, interest income becomes a larger source of overall income inequality when both age-dependent features are removed and the lifespan is extended.

Because of risk aversion, agents dislike inequality and, therefore, policy can improve welfare by reducing this income inequality. In the life cycle and counterfactual models, moving from public debt to public savings increases the wage

[24]To construct inequality measures, we use lifetime asset income as a share of lifetime total income: $\sum_{j=1}^{J} s_j \left(\frac{1}{1+r}\right)^{j-1} ra_j \ / \ \left(\sum_{j=1}^{J} s_j \left(\frac{1}{1+r}\right)^{j-1} we_j h_j + \sum_{j=1}^{J} s_j \left(\frac{1}{1+r}\right)^{j-1} ra_j\right)$. For the No Age-Dependent Features and Long Lifespan counterfactual models, there is no mortality risk so that $s_j = 1$ for all $j = 1, \ldots, J$.

Figure 3: Cumulative Distribution Function for Wealth



Notes: Solid lines represent the baseline economy and dashed lines represent economies with optimal policy.

and decreases the interest rate. This increases lifetime income inequality from savings and decreases lifetime income inequality from labor earnings. Thus, optimal policy must weigh this trade-off. Asset income contributes more to lifetime total income inequality when age-dependent features are removed and lifespan is extended. Accordingly, shifting toward a higher level of public savings will reduce lifetime total income inequality. The change in the wealth distribution in Figure 3 and the total lifetime income inequality measures in Table 4(b) demonstrate that, in fact, adopting an optimal public savings policy reduces both lifetime asset income and total income inequality. Thus, overall, eliminating age-dependent features and extending the lifespan both cause an increase in the optimal level of public savings due to the inequality channel.

Finally, the primary difference between the Long Life counterfactual model and the infinitely lived agent model is the existence of an accumulation phase. Despite their other common features (e.g., no age-dependent features, and long or infinite lifetimes), the infinitely lived agent model features a starkly different optimal policy of public debt as opposed to public savings. As Section 2 explained, the existence of an accumulation phase mitigates the efficacy of the insurance channel, while extending agents' working lifetime further enforces the inequality channel. Thus, when comparing the life cycle and the infinitely lived agent models, the existence of age-dependent features and a shorter lifespan drive optimal policy toward public debt, while the existence of the accumulation phase drives optimal policy toward public savings. Overall, we find that the effects of the accumulation phase dominate the effects of other life cycle model features on optimal policy, thereby ultimately resulting in the optimality of public savings.

5.3 Welfare Decomposition

While the infinitely lived agent model prescribes that the government be a net debtor, the life cycle model's optimal policy prescribes being net saver. What is the welfare loss from incorrectly implementing a public debt policy?

We quantify the welfare consequence of ignoring life cycle features and, as a consequence, adopting a public debt instead of a public savings policy. To do so, suppose that the government implements the optimal debt policy from an infinitely lived agent economy when the true economy is a life cycle economy. We then measure the welfare loss from implementing a suboptimal debt policy using *consumption equivalent variation (CEV)*. CEV is the percent of lifetime consumption that an agent would be willing to pay, ex ante, in order to live in an economy with an optimal public savings policy instead of a suboptimal public debt policy.

Table 5 reports the consumption equivalent variation. We find that an 80 percentage point difference in fiscal policy corresponds to a welfare loss of 0.45 percent of expected lifetime consumption. The welfare loss is economically significant, demonstrating that ignoring life cycle features when determining optimal debt policy will have nontrivial welfare effects. The same 80 percentage point change to government policy in the infinitely lived agent model leads to much smaller welfare effects. Specifically, an infinitely lived agent would only sacrifice 0.04 percent of lifetime consumption in order to live in the economy in which the government has optimal debt instead of 59 percent of output in public savings.

The welfare gains from implementing optimal policy reflect the government's desire to improve the aggregate resources available to agents and the allocation

Table 5: Welfare Decompositions

	Life Cycle (% Change)	**Infinitely Lived (% Change)**
Overall CEV	0.45	-0.05
Level (Δ_l)	0.51	-0.16
Consumption (Δ_{C_l})	1.42	0.91
Hours (Δ_{H_l})	-0.90	-1.06
Distribution (Δ_d)	-0.07	0.10
Consumption (Δ_{C_d})	0.08	-0.20
Hours (Δ_{H_d})	-0.15	0.30

Notes: The life cycle and infinitely lived agent model welfare decompositions compare allocations under a 24% public debt-to-output and a 59% public savings-to-output ratio.

of those resources across agents. In order to characterize these welfare effects, we decompose the consumption equivalent variation (denoted Δ_{CEV}) into a *level effect* (Δ_l) and a *distribution effect* (Δ_d) as follows:[25]

$$(1+\Delta_{CEV}) = [\underbrace{(1+\Delta_{C_l})(1+\Delta_{H_l})}_{\equiv(1+\Delta_l)}] \cdot [\underbrace{(1+\Delta_{C_d})(1+\Delta_{H_d})}_{\equiv(1+\Delta_d)}].$$

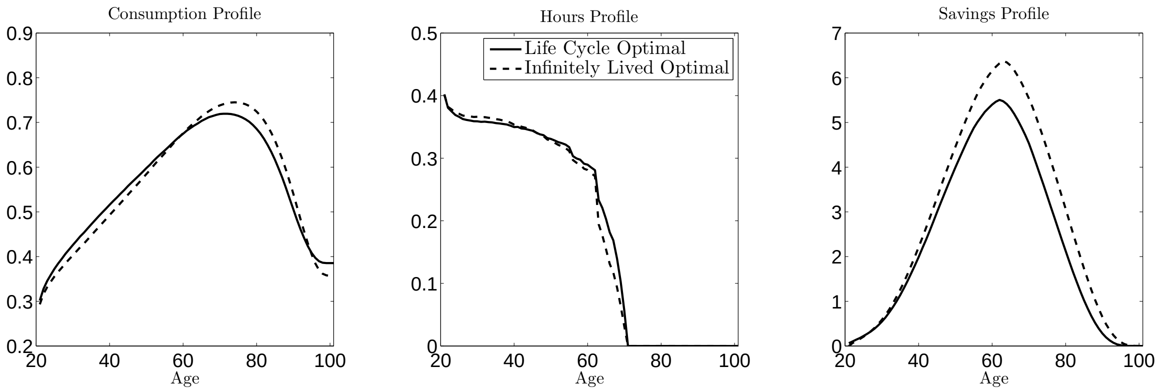
The level effect measures the average agent's change in welfare as a result of changes in aggregate consumption (Δ_{C_l}) and aggregate hours (Δ_{H_l}). The level effect captures the welfare change for a fictitious "representative agent," absent distributional concerns of policy. On the other hand, the distribution effect measures the remaining change in welfare that results from a change in the distribution of consumption (Δ_{C_d}) and hours (Δ_{H_d}) across agents.[26]

Adopting public savings has differential welfare effects across the two mod-

[25]More generally, we follow Floden (2001) in characterizing four components of the CEV: a level effect (Δ_L), an insurance effect (Δ_I), a redistribution effect (Δ_R) and a labor hours effect (Δ_H). We combine the insurance and redistribution effects to form the "distribution effect". Appendix A.2 formally derives the decomposition.

[26]Note that the life cycle model only assumes initial heterogeneity with respect to the permanent and transitory components of labor productivity, but not initial wealth heterogeneity. While allowing for heterogeneity in the initial wealth distribution could generate a larger distribution effect in welfare changes, the PSID and SCF document low levels and relatively small dispersion in individuals' wealth upon entering the labor market.

Figure 4: Life Cycle Model: Consumption, Savings and Hours Profiles



Notes: Solid lines are cross-sectional averages for consumption, savings, and hours by age in the life cycle economy under its optimal public savings policy. The dashed lines are cross-sectional averages for the suboptimal debt policy from the infinitely lived agent economy.

els. Table 5 reports that the 0.45 percent welfare improvement from adopting public savings in the life cycle model can be decomposed into a 0.51 percent increase from the level effect and a partially offsetting 0.07 percent decrease from the distribution effect. The opposite holds for adopting public savings in the infinitely lived agent model, where the 0.04 percent welfare loss corresponds to a 0.70 percent decrease in the level effect and 0.66 percent increase from the distribution effect. These differences in level and distribution effects reflect the varying welfare impact of competing mechanisms across models.

The level effect reflects a difference in the efficacy of the insurance channel across models. To see this, first note that adopting public savings induces a higher wage and a lower interest rate. The higher wage encourages additional labor hours, which increases the resources available for agents' consumption (as seen in Figure 4 for the life cycle model) but worsens total disutility from hours worked. However, Table 5 reports that the percent change in utility from increased aggregate consumption is higher in the life cycle model (1.42) than in the infinitely lived agent model (0.90), despite a larger percent increase in labor disutility in the latter. To account for the relatively smaller consumption increase in the infinitely lived agent model, recall that the lower interest rate discourages private savings in each model. In the infinitely lived agent model, however, the policy also reduces *ex ante* average wealth. This is because, by the nature of living infinite lifespans, aggregate savings is equivalent to ex ante wealth.

Therefore, the lower interest rate worsens an infinitely lived agent's ex ante self-insurance and a larger fraction of the population must sacrifice consumption due to binding liquidity constraints.[27] In contrast, the lower interest rate has no effect on initial allocations in the life cycle model because initial wealth is zero and does not respond to prices.[28]

Finally, the distribution effect corresponds to the inequality channel in both models. Yet, a higher wage has different effects on inequality in the life cycle and infinitely lived agent models. In the life cycle model, since labor earnings contribute more to total income inequality than asset income does, a higher wage exacerbates existing lifetime total income inequality. In the infinitely lived agent model, the opposite holds true: since asset income contributes more to total income inequality, a higher wage alleviates existing lifetime total income inequality. Accordingly, the negative distribution effect in the life cycle model reflects greater income inequality while the positive distribution effect in the infinitely lived agent model reflects lower income inequality.

6 Robustness

In this section we check how sensitive our main results are along three dimensions of the models: matching aggregate labor supply instead of aggregate hours, allowing agents to borrow, and including a more realistic distribution of wealth. We find that our results are robust to each of these model changes.

6.1 Aggregate Labor Supply

In Section 4, we calibrated the coefficient on the disutility of labor in both models, χ_1, to ensure that agents choose to work one third of their *available time endowment*, on average. In the life cycle model, we define this endowment as agents' available time prior to the normal retirement age. However, because agents do not retire in the infinitely lived agent model, the available time endowment is larger and leads to an approximately 25% larger aggregate labor

[27] Relative to Floden (2001), the effect of policy on insurance is usually defined as a distribution effect. However, in order to compare model outcomes, we measure it as an average effect on consumption an hours that is captured by the level effect.

[28] The lower interest rate may decreases average savings across the lifetime, however it does not have as large of a welfare effect since it does not change the initial distribution.

supply under the baseline calibration (see Table 2). Our baseline calibration, while standard, implicitly treats the difference in effective labor supply across models as a defining characteristic of the life cycle model.

In order to test whether this difference in aggregate labor is important for our results, we recompute optimal policy in the infinitely lived agent model under an alternative calibration that makes aggregate labor supply consistent across the two models. Specifically, we now calibrate χ_1 in the infinitely lived agent model to match 27.8 percent, which is the percent of the *total lifetime time endowment* that agents work in the life cycle model.[29] The recalibration reduces the infinitely lived agent model's baseline aggregate labor from $L = 0.66$ to $L = 0.55$, which is quite close to that in the life cycle model.[30]

Under this recalibration, optimal public debt in the infinitely lived agent model equals 31 percent of output. The alternative calibration leads to a fairly small, 10% increase in optimal debt compared to the benchmark calibration. Therefore, changing the calibration target for labor supply does not alter the result that adding a life cycle has large effects on optimal policy.

6.2 Borrowing

In Section 4, we assumed that agents could not borrow. However, the insurance channel operates by endogenously relaxing borrowing constraints and, furthermore, more strongly impacts optimal policy in the infinitely lived agent model. In order to test how sensitive the difference in optimal policy (and, subsequently, the insurance channel) across models is to the no borrowing assumption, we exogenously relax borrowing constraints in both models and recompute optimal policy. Specifically, we set a new borrowing limit at 30 percent of each economy's aggregate private savings under the baseline calibration, while holding all other model parameters fixed.

We find that it is still optimal for the government in the infinitely lived agent model to be a net borrower with public debt equal to 24 percent of output. In

[29] Using this target, we find that $\chi_1 = 72.6$ and $\beta = 0.967$.

[30] By construction, aggregate hours are the same in both models under the baseline calibration. However, small differences remain in aggregate labor. This is because although the average hours decisions are the same, the distribution of hours across the productivity distributions are different in the two models. This is partly because of the age-specific human capital in the life cycle model, which is absent in the infinitely lived agent model.

contrast, we find that it is optimal for the government in the life cycle model to be a net saver with public savings equal to 118 percent of output, which is twice as much public savings as the optimal policy with a no borrowing constraint. Therefore, allowing for borrowing increases the discrepancy between the optimal policies in the two models.

When borrowing is allowed in the life cycle model, young agents who enter the economy with zero wealth and experience a sharply increasing average labor productivity profile, $\{\theta_j\}_{j=1}^{\bar{J}_{ret}}$, over the majority of their working lifetime tend to initially borrow against future income, instead of accumulate savings. Borrowing helps agents intertemporally smooth consumption and insure against idiosyncratic shocks. Agents then wait until later in life, when their labor productivity is expected to be relatively high, to repay their debt and become net savers. Accordingly, higher public savings decreases the interest rate, which simultaneously encourages relatively greater early-life consumption and makes debt repayment less costly.

While life cycle agents' incentives to borrow derive from their increasing average labor productivity profile, infinitely lived agents experience a constant average labor productivity profile, $\theta_j = \bar{\theta}$ for all j. While persistent shocks (ν) in both models generate expected labor productivity growth conditional on having low labor productivity, a constant average labor productivity profile reduces the mass of agents experiencing labor productivity growth. This reduces the average agent's incentive to borrow relative to the life cycle model. As a result, we find a minimal effect on optimal policy from allowing borrowing for these infinitely lived agents. Therefore, the main mechanism by which government debt improves welfare in the infinitely lived agent model is robust to changes in borrowing limits.

7 Conclusion

This paper characterizes the effect of a life cycle on optimal public debt and evaluates the mechanisms by which a life cycle affects optimal policy. We find that the optimal policies are strikingly different between life cycle and infinitely lived agent models. We find that it is optimal for the government to be a *net saver* with savings equal to 59% of output when life cycle features are included.

In contrast, it is optimal for the government to be a *net debtor* with debt equal to 24% of output when these life cycle features are excluded.

Furthermore, there are economically significant welfare consequences from not accounting for life cycle features when determining the optimal policy. We find that if a government implemented the infinitely lived agent model's optimal 24% debt-to-output policy in the life cycle model, then life cycle agents would be worse off by nearly one-half percent of expected lifetime consumption.

We have shown that the existence of the accumulation phase is the predominant reason for the drastically different optimal policies between the life cycle and infinitely lived agent models. In the infinitely lived agent model, higher public debt implies that an average agent begins each period of time with more savings and is, therefore, better insured against labor earnings risk. In the life cycle model, in contrast, agents enter the economy with little or no wealth and must accumulating savings. While a higher level of public debt might encourage life cycle agents to hold more savings during their lifetime, the fact that agents must accumulate this savings stock mitigates the welfare benefits from public debt.

When using quantitative models to answer economic questions, economists constantly face a trade-off between tractability and realism. Our results demonstrate that when examining the welfare consequences of public debt, it is not without loss of generality to utilize the more tractable infinitely lived agent model instead of a life cycle model.

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A Appendix

A.1 Construction of the Balanced Growth Path

We construct the Balanced Growth Path in multiple parts. First we construct the Balanced Growth Path using aggregates from the models. Then, we construct the

Balanced Growth Path using individual agents' allocations. The last two sections develop the Balanced Growth Path for any features unique to the infinitely lived agent or life cycle models.

A.1.1 Aggregate Conditions

Balanced Growth Path: A Balanced Growth Path (BGP) is a sequence

$$\{C_t, A_t, Y_t, K_t, L_t, B_t, G_t\}_{t=0}^{\infty}$$

such that (i) for all $t = 0, 1, \ldots$ $C_t, A_t, Y_t, K_t, B_t, G_t$ grow at a constant rate g,

$$\frac{Y_{t+1}}{Y_t} = \frac{C_{t+1}}{C_t} = \frac{A_{t+1}}{A_t} = \frac{K_{t+1}}{K_t} = \frac{B_{t+1}}{B_t} = \frac{G_{t+1}}{G_t} = 1 + g$$

(ii) per capita variables all grow at the same constant rate g_w:

$$\frac{Y_{t+1}/N_{t+1}}{Y_t/N_t} = \frac{C_{t+1}/N_{t+1}}{C_t/N_t} = \frac{A_{t+1}/N_{t+1}}{A_t/N_t} = \frac{K_{t+1}/N_{t+1}}{K_t/N_t} = \frac{B_{t+1}/N_{t+1}}{B_t/N_t} = \frac{G_{t+1}/N_{t+1}}{G_t/N_t} = 1 + g_w$$

and (iii) hours worked per capita are constant:

$$\frac{L_{t+1}}{N_{t+1}} = \frac{L_t}{N_t} = \frac{L_0}{N_0}$$

Denote time 0 variables without a time subscript, for example $L \equiv L_0$.

Growth Rates: Let all growth derive from TFP $g_z > 0$ and population $g_n > 0$ growth. Then on a balanced growth path we assume:

$$Z_t = (1 + g_z)^t Z$$

$$N_t = (1 + g_n)^t N$$

where z and N are steady state values. Then, from part (iii) of the definition, growth in labor is:

$$\frac{L_{t+1}}{L_t} = \frac{L_{t+1}/N_{t+1}}{L_t/((1 + g_n)N_t)} = 1 + g_n$$

In steady state $Y = ZK^{\alpha}L^{1-\alpha}$. Let output growth be given by $g > 0$. Therefore the production function gives:

$$Y_t = Z_t K_t^{\alpha} L_t^{1-\alpha} \quad \Longrightarrow \quad (1+g) = (1+g_z)^{\frac{1}{1-\alpha}}(1+g_n)$$

Lastly, from parts (ii) and (iii) of the Balanced Growth Path definition, we can solve for the growth of per capita variables:

$$\frac{Y_{t+1}/N_{t+1}}{Y_t/N_t} = \frac{Z_{t+1}}{Z_t}\left(\frac{K_{t+1}/N_{t+1}}{K_t/N_t}\right)^{\alpha}\left(\frac{L_{t+1}/N_{t+1}}{L_t/N_t}\right)^{1-\alpha} \Longrightarrow (1+g_w) = (1+g_z)^{\frac{1}{1-\alpha}}$$

Prices: From Euler's theorem we know:

$$Y_t = \alpha Y_t + (1-\alpha)Y_t = (r_t + \delta)K_t + w_t L_t$$

Accordingly, the wage and interest rate depend on the capital-labor ratio. Growth in the capital-labor ratio is:

$$\frac{K_{t+1}/L_{t+1}}{K_t/L_t} = (1+g_z)^{\frac{1}{1-\alpha}} = 1 + g_w$$

Therefore, the growth rate for the wage is:

$$\frac{w_{t+1}}{w_t} = \frac{Z_{t+1}}{Z_t} \cdot \left(\frac{K_{t+1}/L_{t+1}}{K_t/L_t}\right)^{\alpha} = 1 + g_w$$

and the growth rate for the interest rate is:

$$\frac{r_{t+1}+\delta}{r_t+\delta} = \frac{Z_{t+1}}{Z_t} \cdot \left(\frac{K_{t+1}/L_{t+1}}{K_t/L_t}\right)^{\alpha-1} = 1$$

Therefore wages grow while interest rates do not.

Equilibrium Conditions: The detrended *asset market clearing condition* is:

$$K_t = A_t + B_t \quad \Longrightarrow \quad K = A - B$$

The detrended *resource constraint* is:

$$C_t + K_{t+1} + G_t = Y_t + (1-\delta)K_t \quad \Longrightarrow \quad C + (g+\delta)K + G = Y$$

and the detrended *government budget constraint* is:

$$G_t + rB_t = T_t + B_{t+1} - B_t \quad \Longrightarrow \quad G + (r-g)B = T$$

A.1.2 Individual Conditions

Preferences: We assume that labor disutility has a time-dependent component. Specifically, we assume labor disutility grows at the same rate as the utility over consumption, such that $v_{t+1}(h) = (1+g_w)^{1-\sigma} v_t(h)$. Therefore, total utility is:

$$U_t(c_t, h_t) = u(c_t) - v_t(h_t) = \left[(1+g_w)^{1-\sigma}\right]^t (u(c) - v(h)).$$

Social Security: In order for the AIME to grow at the same rate as the wage, we assume a cost of living adjustment (COLA) on Social Security taxes and payments. For social security taxes, the cap on eligible income grows at the rate of wage growth, $\bar{x}_t = (1+g_w)^t \bar{x}$. Furthermore, base payment bend points $b_{i,t}^{ss} = (1+g_w)^t b_i^{ss}$ and base payment values $\tau_{r,i,t} = (1+g_w)^t \tau_{r,i}$ for $i = 1, 2, 3$.

Tax Function: On a Balanced Growth Path, (c_t, a'_{t+1}, a_t) and $\tilde{y}_t$ must all grow at the same rate as the wage. Furthermore, the tax function must grow at the same rate as the wage. Recalling the tax function, $\Upsilon_t(\tilde{y}_t)$, τ_2 must grow at the same rate as $\tilde{y}_t^{-\tau_1}$. Rewrite as:

$$\Upsilon_t(\tilde{y}_t) = \tau_0 \left((1+g_w)^t \tilde{y} - \left([(1+g_w)^t]^{-\tau_1} \tilde{y}^{-\tau_1} + [(1+g_w)^t]^{-\tau_1} \tau_2 \right)^{-\frac{1}{\tau_1}} \right) = (1+g_w)^t \, \Upsilon(\tilde{y})$$

Individual Budget Constraint: An agent's time t budget constraint is:

$$c_t + a'_{t+1} \leq w_t \varepsilon_t h_t + (1+r_t) a_t - T_t(\cdot)$$
$$c + (1+g_w) a' \leq w \varepsilon h + (1+r) a - T(\cdot)$$

where $\{c, a', a, h, w, r, \varepsilon\}$ are stationary variables. Given that the tax function $\Upsilon(\tilde{y})$ grows at rate g_w, so will the transfer function $T(h, a, \varepsilon)$ in the infinitely lived agent model. Furthermore, given that the Social Security program $\{\bar{x}, b_i^{ss}, \tau_{r,i}\}$ grows at rate g_w, so will the transfer $T(h, a, \varepsilon, x, \zeta')$ function in the life cycle model.

A.1.3 Life Cycle Model

Individual Problem: On the balanced growth path of the life cycle model, the stationary dynamic program is:

$$V_j(a,\varepsilon,x,\zeta) = \max_{c,a',h,\zeta'} \left[u(c) - \zeta' v(h)\right] + \left[\beta s_j (1+g_w)^{1-\sigma}\right] \sum_{\varepsilon'} \pi_j(\varepsilon'|\varepsilon) V_{j+1}(a',\varepsilon',x',\zeta')$$

$$\begin{aligned} \text{s.t.} \quad c + (1+g_w)a' &\leq \zeta' w e(\varepsilon) h + (1+r)(a+Tr) + T(h,a,\varepsilon,x,\zeta') \\ a' &\geq \underline{a} \\ \zeta' &\in \{\mathbb{1}(j < \underline{J}_{ret}),\ \mathbb{1}(j \leq \bar{J}_{ret}) \cdot \zeta\} \end{aligned}$$

Distributions: For j-th cohort at time t, the measure over (a,ε,x,ζ) is given by:

$$\begin{aligned} \lambda_{j,t}(a_t,\varepsilon,x_t,\zeta) &= \lambda_{j,t-1}\left(\frac{a_t}{1+g_w},\varepsilon,\frac{x_t}{1+g_w},\zeta\right)(1+g_n) \\ &= \lambda_{j,t-m}\left(\frac{a_t}{(1+g_w)^m},\varepsilon,\frac{x_t}{(1+g_w)^m},\zeta\right)(1+g_n)^m \quad \forall\, m \leq t \\ &= \lambda_j(a,\varepsilon,x,\zeta) N_{t-j+1}. \end{aligned}$$

Therefore, $\lambda_j(a,\varepsilon,x,\zeta)$ is a stationary distribution over age j agents that integrates to one.

Aggregation: Aggregate consumption in the life cycle model is constructed as follows. Define the relative size of cohorts as $\mu_1 = 1$ and:

$$\mu_{j+1} = \frac{N_{t-j}}{N_t} \cdot \prod_{i=1}^{j} s_i = (1+g_n)^{-j} \prod_{i=1}^{j} s_i = \frac{s_j \mu_j}{1+g_n} \quad \forall\, j = 1,\ldots,J-1$$

Let $C_{j,t}$ be aggregate consumption per age-j agent, which is derived from the age-j agent's allocation:

$$C_{j,t} = \int (1+g_w)^t c_j(a,\varepsilon,x,\zeta) d\lambda_j = (1+g_w)^t \int c_j(a,\varepsilon,x,\zeta) d\lambda_j = (1+g_w)^t C_j$$

where C_j is the stationary aggregate consumption per age-j agent. Accordingly, aggregate consumption is:

$$C_t = N_t\Big(C_{1,t} + s_1(1+g_n)^{-1}C_{2,t} + \cdots + \left(\prod_{i=1}^{J-1} s_i\right)(1+g_n)^{-(J-1)}C_{J,t}\Big)$$

$$= (1+g_w)^t N_t \sum_{j=1}^{J} \mu_j C_j$$

$$= (1+g)^t C$$

where C is the stationary level of aggregate consumption and where we have normalized $N = 1$.

We can similarly construct the remaining aggregates $\{A, K, Y, B, G\}$ on the balanced growth path. Notably, however, labor per capita does not grow. Aggregate labor per capita is constructed as:

$$L_t = N_t \sum_{j=1}^{J} \mu_j L_j \quad \Longrightarrow \quad L = \frac{L_t}{N_t} = \sum_{j=1}^{J} \mu_j \int \zeta_j'(a, \varepsilon, x, \zeta)\varepsilon h_j(a, \varepsilon, x, \zeta) d\lambda_j$$

which is the sum over ages of aggregate labor per age-j agent.

A.1.4 Infinitely Lived Agent Model

In order to isolate the effects on optimal policy due to fundamental differences in the life cycle and infinitely lived agent models, and not due to differences in balanced growth path constructs, we want sources of output growth (e.g. TFP and population growth) to be consistent across models. Thus, we incorporate population growth into the infinitely lived agent model. To be consistent with the life cycle model, we construct a balanced growth path in which the infinitely lived agent model's income and wealth distributions grow homothetically. Our representation of this growth concept is consistent with a dynastic model in which population growth arises from agents producing offspring and valuing the utility of their offspring.

To elaborate in more detail, two additional assumptions admit a balanced growth path with population growth. First, agents exogenously reproduce at rate g_n and next period's offspring are identical to each other. Second, the parent values each offspring identically, and furthermore values each offspring as much as they value their self. Formally, if the parent has continuation value $\beta\mathbb{E}[v(a', \varepsilon')]$, then the parent values all its offspring with total value of $g_n\beta\mathbb{E}[v(a', \varepsilon')]$.

These two assumptions imply two features. First, each offspring is identical to its parent. That is, if the parent's state vector is (a', ε') next period, then so is each offspring's state vector. As a result, the value function of each offspring upon birth is $v(a', \varepsilon')$. Second, since the parent values each offspring equal to its own continuation value, it is optimal for the parent to save save $(1 + g_n)a'$ in total. The portion $g_n a'$ is bequeathed to offspring, and the portion a' is kept for next period.

Individual Problem: On the balanced growth path of the Infinitely Lived Agent Model, the stationary dynamic program is then:

$$v(a, \varepsilon) = \max_{c,a',h} U(c, h) + [\beta(1 + g_w)^{1-\sigma}](1 + g_n) \sum_{\varepsilon'} \pi(\varepsilon'|\varepsilon)v(a', \varepsilon')$$

$$\text{s.t.} \qquad c + (1 + g_n)(1 + g_w)a' \leq w\varepsilon h + (1 + r)a - T(y)$$

where $y \equiv w\varepsilon h + ra$ and optimality conditions are given by:

$$\chi v(h) = u'(c)w\varepsilon(1 - T'(y))$$

$$u'(c) = \beta(1 + g_w)^{-\sigma}(1 + r) \sum_{\varepsilon'} \pi(\varepsilon'|\varepsilon)u'(c')(1 - T'(y')).$$

Notice that the optimality conditions do not change relative to a world without population growth. However, the cost of savings has increased since agents bequeath wealth to offspring.

Distribution: The distribution evolves according to:

$$\lambda_{t+1}(a_{t+1}, \varepsilon_{t+1}) = \sum_{\varepsilon_t} \pi(\varepsilon_{t+1}|\varepsilon_t) \int_A \mathbb{1}\left[a'_{t+1}(a_t, \varepsilon_t) = a_{t+1}\right] \lambda_t(a_t, \varepsilon_t)da_t$$

The stationary distribution $\lambda(a, \varepsilon)$ has measure 1 over $\mathcal{A} \times \mathcal{E}$ but the mass of agents grows at rate g_n:

$$\lambda_t(a_t, \varepsilon) = \lambda_{t-1}\left(\frac{a_t}{1 + g_w}, \varepsilon\right)(1 + g_n)$$

$$= \lambda_{t-s}\left(\frac{a_t}{(1+g_w)^s}, \varepsilon\right)(1+g_n)^s \quad \forall\, s \leq t$$

$$= \lambda(a,\varepsilon)N_t$$

Therefore, applying the transformation above and normalizing by N_{t+1} yields:

$$\lambda(a',\varepsilon') = \sum_{\varepsilon}\pi(\varepsilon'|\varepsilon)\int_A \mathbb{1}\left[a'(a,\varepsilon) = a'\right]\frac{\lambda(a,\varepsilon)}{1+g_n}da$$

Aggregation: To construct aggregate consumption, wealth, savings and labor, multiply individual allocations by the size of the population (N_t) and sum using the stationary distribution λ. For example, aggregate consumption is:

$$C_t = N_t\int(1+g_w)^t c(a,\varepsilon)d\lambda = (1+g)^t\int c(a,\varepsilon)d\lambda = (1+g)^t C$$

We can similarly construct the remaining aggregates $\{A, K, Y, B, G\}$ on the balanced growth path. Notably, however, aggregate labor per capita does not grow:

$$\frac{L_t}{N_t} = \int \varepsilon h(a,\varepsilon)d\lambda$$

where again $N_0 = 1$ by normalization.

A.2 Welfare Decomposition

Proposition 1: *If preferences are additively separable in utility over consumption, $u(c)$, and disutility over hours, $v(h)$, then welfare changes can be decomposed as:*

$$(1+\Delta_{CEV}) = [\underbrace{(1+\Delta_{C_l})(1+\Delta_{H_l})}_{\equiv(1+\Delta_l)}]\cdot[\underbrace{(1+\Delta_{C_d})(1+\Delta_{H_d})}_{\equiv(1+\Delta_d)}].$$

Proof: Consider two economies, $i \in \{1,2\}$. Define ex ante welfare in economy $i \in \{1,2\}$ as:

$$S^i = S^i_c + S^i_h \equiv \int \mathbb{E}_0\left[\sum_{j=1}^{J}\beta^{j-1}s_j u\left(c^i_j\right)\right]d\lambda^i_1 + \int \mathbb{E}_0\left[\sum_{j=1}^{J}\beta^{j-1}s_j v\left(h^i_j, \zeta^i_j\right)\right]d\lambda^i_1.$$

Denote the Consumption Equivalent Variation (CEV) by Δ_{CEV}, which is defined as the percent of expected lifetime consumption that an agent inhabiting economy $i = 1$ would pay *ex ante* in order to inhabit economy $i = 2$:

$$(1 + \Delta_{CEV})^{1-\sigma} S_c^1 + S_h^1 = S^2.$$

Furthermore, define an individual's certainty equivalent consumption as the level $\bar{c}(a, \varepsilon, x, \zeta)$ such that the individual is indifferent between consuming $\bar{c}(a, \varepsilon, x, \zeta)$ at every age with certainty and consuming according to policy function $\{c_j(a, \varepsilon, x, \zeta)\}_{j=1}^J$ with uncertainty. That is, $\bar{c}(a, \varepsilon, x, \zeta)$ is defined by:

$$S_c^i \equiv \int \mathbb{E}_0 \left[\sum_{j=1}^J \beta^{j-1} s_j u\left(c_j^i\right) \right] d\lambda_1^i = \left(\sum_{j=1}^J \beta^{j-1} s_j \right) \int u\left(\bar{c}^i(a_1, \varepsilon_1, x_1, \zeta_1)\right) d\lambda_1^i$$

which implies the definition of aggregate certainty equivalent consumption:

$$\bar{C}^i \equiv \int \bar{c}_1^i(a_1, \varepsilon_1, x_1, \zeta_1) d\lambda_1^i$$

Therefore, if agents only consume their certainty equivalent consumption allocation, then they only face ex ante risk in their consumption. Following Floden (2001), define the *redistribution effect* by a comparison between consuming an individual and aggregate certainty equivalent consumption allocation:

$$\int \mathbb{E}_0 \left[\sum_{j=1}^J \beta^{j-1} s_j u\left((1 - \omega_R^i)\bar{C}^i\right) \right] d\lambda_1^i = \left(\sum_{j=1}^J \beta^{j-1} s_j \right) \int u\left(\bar{c}^i(a_1, \varepsilon_1, x_1, \zeta_1)\right) d\lambda_1^i$$

which implies:

$$1 - \omega_R^i = \frac{(S_c^i / \sum_{j=1}^J \beta^{j-1} s_j)^{\frac{1}{1-\sigma}}}{\bar{C}^i} \quad \text{and} \quad 1 + \Delta_{C_R} \equiv \frac{1 - \omega_R^2}{1 - \omega_R^1} = \frac{(S_c^2 / S_c^1)^{\frac{1}{1-\sigma}}}{\bar{C}^2 / \bar{C}^1}.$$

Likewise, define the *uncertainty effect* as a comparison between consuming at each age, the aggregate consumption allocation:

$$C^i = \sum_{j=1}^J \mu_j \int c_j^i(a, \varepsilon, x, \zeta) d\lambda_j^i$$

and the aggregate certainty equivalent consumption, $\bar{C}^i$. Then:

$$\int \mathbb{E}_0 \left[\sum_{j=1}^{J} \beta^{j-1} s_j u \left((1-\omega_I^i) C^i \right) \right] d\lambda_1^i = \left(\sum_{j=1}^{J} \beta^{j-1} s_j \right) \int u \left(\bar{C}^i \right) d\lambda_1^i$$

which implies:

$$1 - \omega_I^i = \frac{\bar{C}^i}{C^i} \quad \text{and} \quad 1 + \Delta_{C_I} = \frac{1-\omega_I^2}{1-\omega_I^1}$$

Lastly, define the labor disutility effect Δ_H as the percent of lifetime consumption that an individual would pay to change their hours allocation:

$$(1+\Delta_H)^{1-\sigma} S_c^2 = S_c^2 + (S_h^2 - S_h^1)$$

Proceeding from the definition of the CEV, we can decompose welfare as follows:

$$\begin{array}{ccccccccc} (1+\Delta_{CEV}) & = & (1+\Delta_{C_l}) & \cdot & (1+\Delta_{C_I}) & \cdot & (1+\Delta_{C_R}) & \cdot & (1+\Delta_H) \\ \left(\dfrac{S^2 - S_h^1}{S_c^1} \right)^{\frac{1}{1-\sigma}} & = & (C^2/C^1) & \cdot & \dfrac{\bar{C}^2/\bar{C}^1}{C^2/C^1} & \cdot & \dfrac{(S_c^2/S_c^1)^{\frac{1}{1-\sigma}}}{\bar{C}^2/\bar{C}^1} & \cdot & \dfrac{((S^2 - S_h^1)/S_c^1)^{\frac{1}{1-\sigma}}}{(S_c^2/S_c^1)^{\frac{1}{1-\sigma}}} \end{array}$$

Canceling terms on the right hand side of the expression readily shows a decomposition holds as desired. In the text, we combine $(1+\Delta_{C_I})(1+\Delta_{C_R})$ as an amalgam term, $(1+\Delta_{C_d})$, consistent with Conesa et al. (2009), to form the consumption distribution effect.

Decomposing the labor disutility effect into level and distribution effects follows similar reasoning. Define the hours level effect $(1+\Delta_{H_l})$ as the labor disutility that an agent would be willing to accept in order to work H^2 hours each period instead of H^1 hours, which we express as a first order approximation about Δ_{H_l},

$$\Delta_{H_l} \approx \frac{1+1/\gamma}{1-\sigma} \left(1 - \frac{H^2}{H^1} \right) \frac{S_h^1 + \int \mathbb{E}_0 [\sum_{j=1}^{J} \beta^{j-1} s_j \zeta_j^1 \chi_2] d\lambda_1^1}{S_c^2}.$$

Finally, define the hours distribution effect as the residual of the labor disutility effect after accounting for the hours level effect. Therefore the decomposition gives $(1+\Delta_H) = (1+\Delta_{H_l})(1+\Delta_{H_d})$. ■