Coordination with timing frictions and payoff heterogeneity*

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Abstract

We study a dynamic model of coordination with timing frictions and payoff heterogeneity. Flow payoffs depend on a stochastic fundamental, on others' behavior and on idiosyncratic preferences. Agents get opportunities to revise their behavior according to a Poisson clock. The unique equilibrium is characterized by a threshold that determines the choices for each type of agent. We provide analytical solutions for a particular case with linear preferences and very slow moving fundamentals. A lot of conformity emerges: despite payoff heterogeneity, agents' thresholds partially coincide as long as there exists a set of beliefs that would make this coincidence possible – though they never fully coincide. Moreover, agents' choices are more similar when there is more heterogeneity in their behavior (determined by previous choices). Conformity is not inefficient. The efficient solution would have agents following others even more often and giving less importance to the fundamental.

Keywords: coordination, conformity, timing frictions, payoff heterogeneity.

Jel Classification: C73, D84.

1 Introduction

Strategic complementarities arise in many dynamic settings. Profitability of investment decisions might depend on future demand, which depends on whether others will invest as well. Adopting a new technology may not be the best decision if others in the production chain will keep working with an old technology. Likewise, choosing a Betamax VCR is not a good idea if everyone else plans to buy a VHS VCR. The payoff from joining a social network depends on whether others are expected to join it (or leave it) in the near future. Other examples include bank runs, currency speculation, riots and political upheavals.

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Frankel and Pauzner (2000) provide a dynamic framework that captures some essencial features of these economic problems. Agents make a binary choice between two actions (say joining Facebook or not). The payoff from their decision depends on an exogenously moving fundamental (say the quality of Facebook) and on how many others are on Facebook. Agents get an opportunity to change behavior (join or leave Facebook) according to a Poisson clock, which captures in a simple way the idea that consumers are not thinking about it at every point in time (likewise, people are not buying VCRs, thinking about the financial situation of commercial banks or making choices on investment at every point in time).[1]

All those models assume (ex-ante) identical agents. In consequence, at a point in time, all agents are making the same decision. However, some important questions in those settings are related to the interaction between payoff heterogeneity and dynamic coordination motives. For example, we cannot use this model of technology adoption to talk about early adopters and snowball effects. A model where the fraction of firms willing to invest at a given point in times is either 0 or 1 has little hope of matching the data on investment. Likewise, a model where the fraction of people choosing Facebook over Google+ at a given point in time is either 0 or 1 will not be able to say much about the dynamics of agents' behavior.

The contribution of this paper is the extension of the Frankel-Pauzner model to a setup with heterogeneous agents. First, we show there is a unique rationalizable equilibrium, agents of a given type play according to a threshold that depends on the total number of agents in a network and on the exogenous fundamental. We then provide an analytical characterization for the equilibrium threshold in a tractable case with two types, linear utility and vanishing shocks. Last, we solve the planner's problem to understand the inefficiencies that arise in equilibrium.

Each type of agent joins the network if the exogenous fundamental (θ) is larger than a threshold that is a function of the fraction of agents in the network (n). In the tractable case with vanishing shocks, a lot of conformity arises: different types will always play the same strategy for some values on n unless their preferences are so heterogeneous that there is no set of beliefs that would induce them to play according to the same threshold. Another interesting result is that agents' choices are more similar for intermediate values of n, when there is more heterogeneity in their behavior.

[1] Frankel and Pauzner (2000) base their analysis on a model of sectorial choice (along the lines of Matsuyama (1991)), but their framework has been used to analyse location choices (Frankel and Pauzner (2002)), carry trades and speculation (Plantin and Shin (2006)), speculative attacks (Daniëls (2009)) and investment and business cycles (Frankel and Burdzy (2005), Guimaraes and Machado (2014)). The model of currency attacks in Guimaraes (2006) and the model of debt runs in He and Xiong (2012) employ similar timing frictions.

We then compare the solution to the central planner's problem and to the decentralized equilibrium. The planner also acts according to thresholds that are specific to each type of agent. For the case with 2 types, linear utility and vanishing shocks, we provide an analytical characterization of the planners' threshold.

The planner chooses even more conformity in behavior than the agents. The range where idiosyncratic tastes are relevant for the planner is smaller than the analogous range for agents in the decentralized equilibrium. The reason is that the planner internalizes the externalities on others, which are the same for all agents.

Interestingly, the planner also gives less importance to the exogenous fundamental than the agents. In a problem of two-sided network externalities, that means the planner will give less importance to the intrinsic quality of each good than the agents. For example, agents tend to follow the crowd and choose, say, VHS over Betamax because everybody else has a VHS VCR, even if Betamax is better in terms of its intrinsic quality. The planner would be even more inclined towards VHS. Intuitively, the planner cares about everything the agents care about plus the externalities on others. That depends fundamentally on the amount of people in each network (current and expected). The effect of the intrinsic quality of the good is internalized by the agents.

In a symmetric example with two-sided externalities, the planner's solution and the decentralized equilibrium only coincide when there is no heterogeneity in tastes and timing frictions vanish (the frequency agents get to choose which network to join goes to infinity). Asymmetries add another source of inefficiency.

The paper is related to the literature on coordination in games with strategic complementarities. With complete information and no shocks, multiple self-fulfilling equilibria might arise in those settings. Carlsson and Van Damme (1993) and Morris and Shin (1998) have shown that a unique equilibrium arise in a static environment in which fundamentals are not common knowledge and agents have some idiosyncratic information about them. Frankel and Pauzner (2000) and Burdzy et al. (2001) show that a small amount of shocks in a dynamic model (with no private information) yields similar results. The relation between both literatures is discussed in Morris (2014).[2] In a related contribution, Herrendorf et al. (2000) show that if there is enough heterogeneity, there is a unique equilibrium even in a static setting with complete information.

The paper is also related to literature on network externalities, in which strategic complementarities arise from consumption externalities.[3] Agents' optimal choices typically depend on what they expect others will do. However, most of this literature makes ad-hoc

[2]See also Morris and Shin (2003).

[3]This literature has started with Katz and Shapiro (1985) and Katz and Shapiro (1986). See Shy (2011) for a survey.

assumptions on how agents coordinate.[4]. One important exception in Argenziano (2008). She studies welfare in a model with differentiated networks in a static global-game model and highlights two sources of inefficiencies: agents give too much importance to their own idiosyncratic tastes and firms with the larger network charge a higher price. Both effects contribute to make the network "too balanced". Our work complements her work by pointing out inefficiencies coming from the dynamic interaction among agents.[5]

The efficiency results here contrast with those in models with information externalities that generate herd behavior (e.g., Bikhchandani et al. (1992)). In those models, agents follow others too much from a social point of view. Here, conformity of behavior arises because of preferences, not through learning, and they follow others too little.

2 The model with heterogeneous preferences

There is a continuum of agents indexed by $i \in [0, 1]$ who must choose between two actions, A and B. The payoff an agent get from choosing either action depends on fundamentals, on the actions of others,[6] and on the agents' own type (players have heterogeneous preferences). Time is continuous. Agents discount the future at a rate ρ and receive chances to revise their actions according to a Poisson process with arrival rate δ.

There are Q types of players. We denote agent i's relative payoff of choosing B by $\pi_{q(i)}(\theta, n)$, where $\theta \in \mathbb{R}$ denotes the fundamentals of the economy, n is the fraction of agents currently commited to action B and $q(i) \in \{1, ..., Q\}$ is agent i's type. $\pi(.)$ is continuous and strictly increasing in both arguments. Let α_q denote the mass of type-q agents in the population and n_q the proportion of type-q agents currently on B. Thus, $n = \sum_{q=1}^{Q} \alpha_q n_q$.

An agent who receive a chance revise her choice at time τ will pick B whenever

$$\mathbb{E} \int_{\tau}^{\infty} e^{-(\rho+\delta)(t-\tau)} \pi_{q(i)}(\theta_t, n_t) dt \geq 0$$

and will pick A otherwise.

We further assume that payoff functions $\pi_q(.)$ are such that there are dominance regions for all types of agents. For each type, there is a region in the $\mathbb{R} \times [0, 1]$ space where choosing A is a dominant action, and a region in which choosing B is a dominant action. In words,

[4]For instance, Katz and Shapiro (1986) assume that whenever there are multiple equilibria in the model, agents manage to coordinate their decisions in order to achieve the Pareto-superior outcome

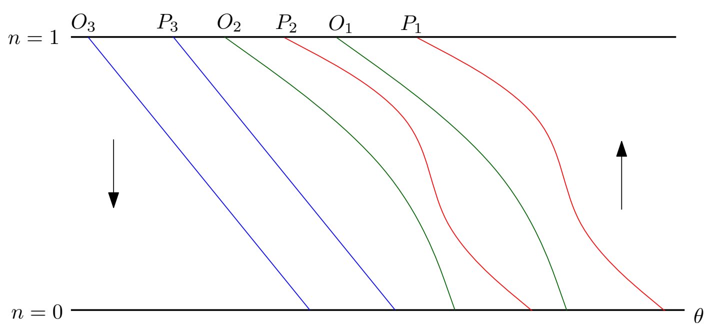
[5]See also Ambrus and Argenziano (2009).

[6]Strategic complementarities can arise owing to either one-sided externalities or two-sided externalities. It can be the case that the payoff of choosing action A is independent of the amount of agents making the same choice, but the payoff of choosing B is increasing in this amount, or it can be the case that both actions become more appealing the larger is the proportion of agents taking them.

there is a sufficient low level of the fundamentals for which an agent prefers to play A even if all other agents are playing B, and there is a sufficient high level of the fundamentals such that it is preferable to play B even if no one else is expected to do it.

Let P_q be the boundary of the upper dominance region of a type-q agent, i.e., the curve on which such agent is indifferent between the two actions if she believes everyone after her will choose A (P stands for *pessimistic* about the proportion of agents playing B in the future). These boundaries are downward sloping: since $\pi_q(\theta, n)$ is increasing in θ and n, a higher n today means that the θ needed to make agents indifferent between the two actions is smaller. At the other extreme, let O_q be the boundary of the lower dominance region for a type-q player, that is, the curve on which this type of agent is indifferent between the two actions under the belief that everyone will choose B when they get the chance (O stands for *optimistic*). These boundaries are also downward sloping.

Figure 1: Dominance regions: an example



2.1 Unique equilibrium

Proposition 1. *If θ is constant over time, there may be multiple equilibria.*

Proof. See Appendix. □

Of course, when θ lies either to the right of all upper dominance regions boundaries, or to the left of all lower dominance regions boundaries, the equilibrium is unique. However, when θ is such that neither action is dominant for some agents, multiplicity may arise, depending on the amount of heterogeneity.[7]

When the fundamental changes stochastically, however, the equilibrium is unique for any amount of heterogeneity. Theorem 1 states this result. The following lemma is key for the demonstration.

[7] Herrendorf et al. (2000) show that, in a similar environment with no shocks, if there is a sufficient amount of heterogeneity multiplicity is ruled out.

Lemma 1. *For any strategy profile, the dynamics of n depends only on (θ_t, n_t). It does not depend on each $n_{q,t}$, $q \in Q$.*

Proof. Fix a strategy profile $\left\{s_{q(i)}\right\}_{q\in\{1,\ldots,Q\}}$. Denote by $\mathcal{B}_t$ the set of types playing B at time t. Notice that the path of n_q is given by the following differential equation:

$$\frac{\partial n_{q,t}}{\partial t} = \begin{cases} \delta(1-n_{q,t}) & \text{if } q \in \mathcal{B}_t \\ -\delta n_{q,t} & \text{if } q \notin \mathcal{B}_t \end{cases} \tag{1}$$

Equation 1 means that, every type-q agent whose strategy prescribes playing B and who is not already on B and receives a chance to revise her choice at time t will switch to B (there are $1 - n_{q,t}$ such agents). On the contrary, every type-q agent whose strategy prescribes playing A and who has previously chosen B will leave it at her first opportunity. Using the fact that $n = \sum_{q=1}^{Q} \alpha_q n_{q,t}$, we have that $\frac{\partial n_t}{\partial t}$ is given by:

$$\begin{aligned} \frac{\partial n_t}{\partial t} &= \sum_{q=1}^{Q}\left(\alpha_q \frac{\partial n_{q,t}}{\partial t}\right) \\ &= \sum_{q\in\mathcal{B}_t} \alpha_q \delta(1-n_{q,t}) + \sum_{q\notin\mathcal{B}_t} \alpha_q\left(-\delta n_{q,t}\right) \\ &= \delta\left[\sum_{q\in\mathcal{B}_t}\alpha_q - \sum_{q=1}^{Q}\alpha_q n_{q,t}\right] \\ \iff \frac{\partial n_t}{\partial t} &= \delta\left[\sum_{q\in\mathcal{B}_t}\alpha_q - n_t\right] \end{aligned}$$

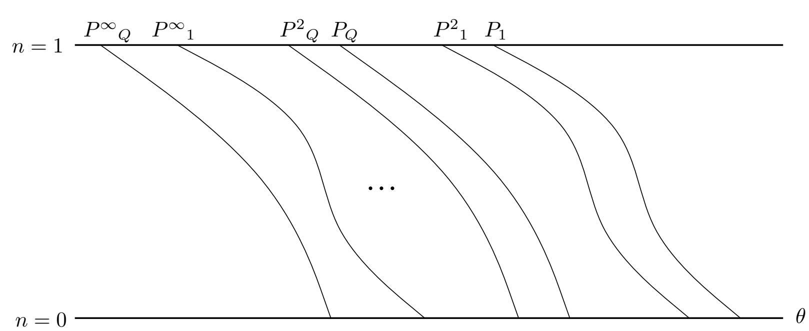
□

Lemma 1 allows us to deal with this problem in a two-dimensional space: agents need only to look at the fundamentals (θ_t) and at the aggregate mass of agents currently on B in order to understand the dynamics of the system. One could expect this dynamics to depend on the proportion of each type of agent currently on each option, but due to the assumption of a Poisson process for the arrival of opportunities to switch actions, that is not true. It suffices to know the aggregate n_t and each type's choices to compute $\partial n_t/\partial t$.

Theorem 1. *Suppose θ follows a Brownian motion with drift μ and variance $\sigma^2 > 0$. There is an unique equilibrium characterized by thresholds $\left(Z_q^*\right)_{q\in\{1,\ldots,Q\}}$ in the $\mathbb{R}\times[0,1]$ space. Each agent i plays B when to the right and A when to the left of $Z_{q(i)}^*$.*

Proof. The proof follows the same reasoning as in the case of identical individuals (Frankel and Pauzner (2000)). The existence of dominance regions give us a starting point to a iterated elimination of strictly dominated strategies procedure. Consider a type-q agent at some point on P_q. She is indifferent between A and B under the belief that everyone choosing networks while she is committed to her choice will pick A under any circumstances. But when θ moves stochastically, there is always the possibility that it will spend some time to the right of some players' dominance region boundaries. Notice that even if q is such that P_q is the leftmost upper boundary (say P_3 in figure (1)), she cannot expect every other player to choose A under any circumstances while she is commited to her choice. If θ moves slightly to the right, it will be strictly dominant for type-q agents to pick B, and thus a fraction α_q of the agents that get the chance will not choose A. The most pessimistic (to network B) belief that agents can hold consistent with the dominance regions is that each type-q agent plays B when to the right of P_q , and A when to the left of it. In other words, agents do not play strictly dominated strategies. Under this (more optimistic) new belief, the agent on P_q is not indifferent anymore, but strictly preferring to join B. To make her indifferent, we must lower θ. We can then construct for each type q a new boundary P_q^2 (to the left of P_q), to the right of which a type-q player chooses B when she expects all other agents to play according to $(P_q)_{q\in\{1,\dots,Q\}}$. This procedure can be repeated *ad infinitum*. At each round, we look for the curve P_q^k on which a type-q player has zero discounted payoff when assuming that other agents play according to $\left(P_q^{k-1}\right)_{q\in\{1,\dots,Q\}}$. Denote the limit of this sequence by $\left(P_q^{\infty}\right)_{q\in\{1,\dots,Q\}}$. Notice that each agent i playing according to $P_{q(i)}^{\infty}$ is, in fact, an equilibrium: if she expects others to play according to $\left(P_q^{\infty}\right)_{q\in\{1,\dots,Q\}}$, her best response is to play according to $P_{q(i)}^{\infty}$.

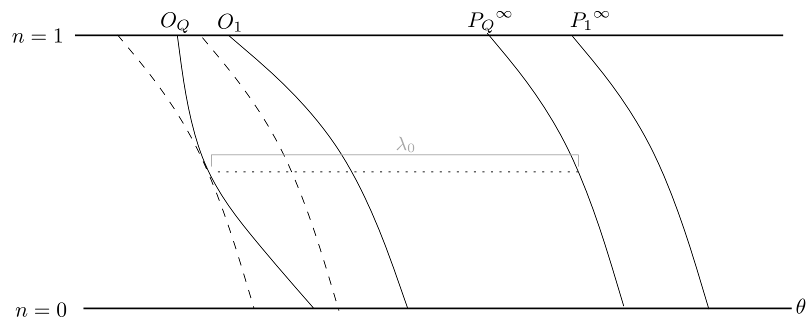
Figure 2: Iterative deletion of strictly dominated strategies from the upper dominance region



We now turn to a different iterative process from the lower dominance regions. Let $\left(P_q^{\lambda_0}\right)_{q\in\{1,\dots,Q\}}$ be translations of the curves P_q^{∞} to the left by an amount λ_0. Fix λ_0 as the smaller distance such that all translations lie completely on the lower dominance region of each corresponding type. Figure (3) below exemplifies this step.

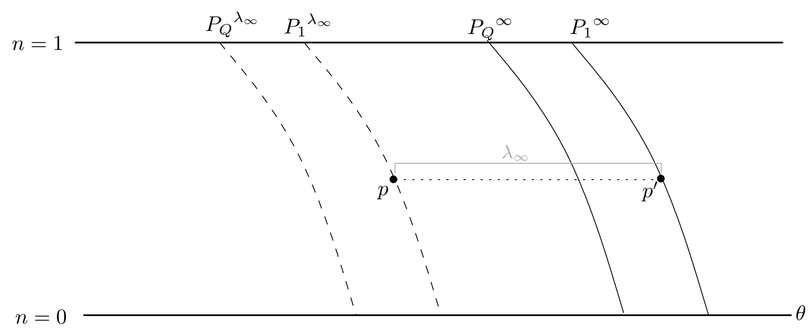
Now, construct for each type a new curve $P_q^{\lambda_1}$ as the rightmost translation of $P_q^{\lambda_0}$ to

Figure 3: Translations of P_q^{∞}



the left of which each type-q agent must play A if they expect others to play according to $\left(P_q^{\lambda_0}\right)_{q\in\{1,\ldots,Q\}}$.[8] Let $P_q^{\lambda_\infty}$ be the limit of this sequence, for each q. There is at least one point in some $P_q^{\lambda_\infty}$ curve on which a type-q agent is indifferent between the two networks, otherwise iterations would not have stopped. Without loss of generality, suppose there is a point of indifference in $P_1^{\lambda_\infty}$ and name it p. Let p' denote the point on P_1^{∞} at the same height as p. If we establish that p and p' coincide, we show that the whole curves coincide and, since we have translated all curves by the same λ's, $P_q^{\lambda_\infty} = P_q^{\infty} \forall q$, that is, the equilibrium is unique.

Figure 4: Equilibrium uniqueness



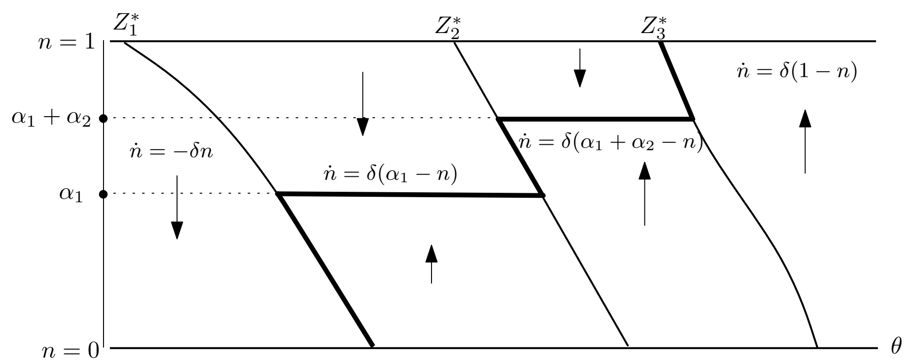
Lets compare two type-1 players, one receiving an opportunity to choose a network on p (expecting others to play according to the limit translations), and the other on p' (expecting others to play according to $\left(P_q^{\infty}\right)_{q\in\{1,\ldots,Q\}}$). Lets name those players p and p', respectively. We know that both players expect changes in the fundamentals relative to its starting point to have the same distribution. Also, since the original curves and their translations have the same shape and the pairwise distances between $P_q^{\lambda_\infty}$'s are the same as the distances between P_q^{∞}'s (each round, we have translated all curves by the same λ), we know by Lemma 1 that, for a given path of the fundamental, they both expect

[8] Note that what we are doing is eliminating strictly dominated strategies once again, but we are not necessarily eliminating *all* dominated strategies each round.

the same dynamics for n_t. If $\lambda_\infty > 0$, we get a contradiction: the two players expect the same relative dynamics for the (θ_t, n_t) system and the θ that p' expects at all times exceeds the θ that the agent on p expects, thus they cannot both have zero payoff. Then, $\lambda_\infty = 0$, that is, the points p and p' must coincide. The equilibrium is unique and it is characterized by threshold $\left(Z_q^*\right)_{q\in\{1,\dots,Q\}}$, where $Z_q^* \equiv P_q^\infty$. $\square$

Figure 5 below exemplifies the dynamics around the equilibrium for the case of three types of agents. $\partial n_t/\partial t$ is computed as in Lemma 1.

Figure 5: Dynamics ($Q = 3$)



3 A tractable particular case

Although we can prove existence and uniqueness of equilibrium, we cannot easily compute the equilibrium thresholds for each type of agent analitically in this general case. For this reason, hereafter we will focus on the limit as $\mu \to 0$ and $\sigma \to 0$, that is, the case in which shocks to the fundamentals vanishes. Also, for the sake of simplicity, we will restrict our attention to a linear functional form for $\pi(\theta, n)$ and analyze the case where $Q = 2$. Hence, the payoff from choosing B is given by:

$$\pi_i(\theta_t, n_t) = \theta_t + \gamma n_t + \varepsilon_i.$$

with

$$\varepsilon_i = \begin{cases} \overline{\varepsilon} & \forall i \in [0, \alpha] \\ \underline{\varepsilon} & \forall i \in (\alpha, 1] \end{cases},$$

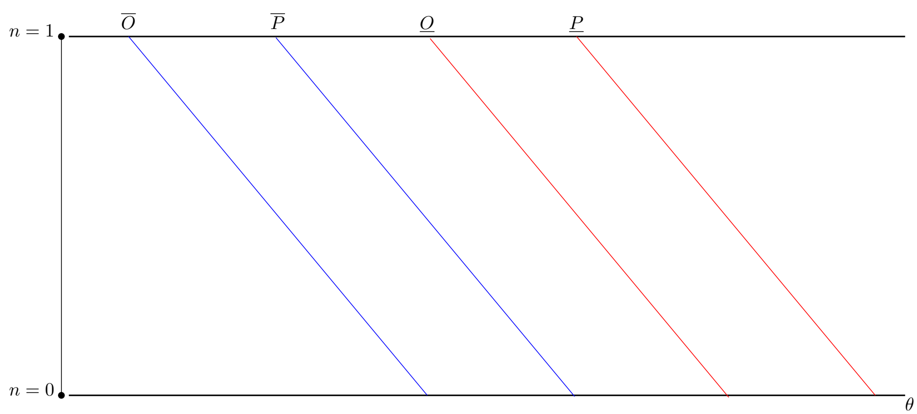
that is, there are two types of agents: a proportion α with preference parameter $\overline{\varepsilon}$ and a proportion $1 - \alpha$ with preference parameter $\underline{\varepsilon}$, $\overline{\varepsilon} > \underline{\varepsilon}$. We will divide the analysis into cases, each one corresponding to a different amount of heterogeneity, which is measured here by the distance between $\overline{\varepsilon}$ and $\underline{\varepsilon}$.

Very Large heterogeneity First we consider the case in which

$$\overline{\varepsilon} - \underline{\varepsilon} > \frac{\gamma\delta}{\rho + 2\delta} \tag{2}$$

This condition ensures that the curve at which a high-type agent with pessimistic beliefs about n is indifferent between A and B is located to the left of the curve at which a low-type agent with optimistic beliefs is indifferent between the two networks. Put differently, $\overline{P} < \underline{O}$ and thus the intersection between the region on which neither action is dominant for a high-type agent and the region with no dominant action for a low-type agent is empty. Figure (6) exemplifies this situation.

Figure 6: Dominance regions when $\overline{\varepsilon} - \underline{\varepsilon} > \frac{\gamma\delta}{\rho+2\delta}$



If the condition in (2) holds, then there is no set of beliefs that could induce different agents to play according to the same threshold $Z^*(n)$ for any value of n. Even with the most pessimistic beliefs (nobody will ever choose B) type-$\overline{\varepsilon}$ agents would still be more inclined to choose B than type-$\underline{\varepsilon}$ agents believing that everybody else would always choose B. Naturally, whenever this condition is satisfied, the equilibrium in the limit as $\mu, \sigma \to 0$ will be such that type-$\overline{\varepsilon}$ and type-$\underline{\varepsilon}$ agents play according to thresholds that do not intersect. Proposition 2 characterizes the equilibrium in this case.

Proposition 2. *Suppose $\overline{\varepsilon} - \underline{\varepsilon} > \frac{\gamma\delta}{\rho+2\delta}$. In the limit as $\mu \to 0$ and $\sigma \to 0$, an agent i called upon choosing a network at time t plays B whenever $\theta_t > Z_i^*(n_t)$ and A otherwise, where Z_i^*'s are computed as follows:*

(To ease notation, let $Z_i^* \equiv \overline{Z}\, \forall i \in [0, \alpha]$ and $Z_i^* \equiv \underline{Z}\, \forall i \in (\alpha, 1]$.)

If $\overline{\varepsilon} - \underline{\varepsilon} \geq \frac{\gamma(\delta+\rho\alpha)}{\rho+2\delta}$**,**

$$\overline{Z} = \begin{cases} -\overline{\varepsilon} - \frac{\gamma(\rho+\delta)}{\rho+2\delta} n & \textit{if } \ n \geq \alpha \\ -\overline{\varepsilon} - \frac{\alpha\gamma\delta}{\rho+2\delta} - \frac{\gamma\rho}{\rho+2\delta} n & \textit{if } \ n < \alpha \end{cases}. \tag{3}$$

If $\overline{\varepsilon} - \underline{\varepsilon} < \frac{\gamma(\delta+\rho\alpha)}{\rho+2\delta}$**,** $\overline{Z}$ *is given by equation (3) above* $\forall n \geq \hat{n}$ *and otherwise it satisfies*

$$\frac{(\alpha-n)}{\alpha} \left\{ \int_{t=0}^{\overline{t}} e^{-(\rho+\delta)t} \left[\overline{Z} + \overline{\varepsilon} + \gamma \left(\alpha - (\alpha-n)e^{-\delta t} \right) \right] dt + \int_{t=\overline{t}}^{\infty} e^{-(\rho+\delta)t} \left(\overline{\varepsilon} - \underline{\varepsilon} \right) dt \right\} + \frac{n}{\alpha} \int_{t=0}^{\infty} e^{-(\rho+\delta)t} \left[\overline{Z} + \overline{\varepsilon} + \gamma n e^{-\delta t} \right] dt = 0, \tag{4}$$

where $\overline{t} = -\frac{1}{\delta} \ln \frac{\alpha + [\left(\overline{Z}+\underline{\varepsilon}\right)(\rho+2\delta)+\gamma\delta]/\gamma(\rho+\delta)}{\alpha-n}$ and $\hat{n} = \alpha - \frac{(\rho+2\delta)(\overline{\varepsilon}-\underline{\varepsilon})-\gamma\delta}{\gamma\rho}$.

If $\overline{\varepsilon} - \underline{\varepsilon} \geq \frac{\gamma[\delta+\rho(1-\alpha)]}{\rho+2\delta}$**,**

$$\underline{Z} = \begin{cases} -\underline{\varepsilon} - \frac{\gamma\delta(1+\alpha)}{\rho+2\delta} - \frac{\gamma\rho}{\rho+2\delta} n & \textit{if } \ n > \alpha \\ -\underline{\varepsilon} - \frac{\gamma\delta}{\rho+2\delta} - \frac{\gamma(\rho+\delta)}{\rho+2\delta} n & \textit{if}\, n \leq \alpha \end{cases}. \tag{5}$$

If $\overline{\varepsilon} - \underline{\varepsilon} < \frac{\gamma[\delta+\rho(1-\alpha)]}{\rho+2\delta}$**,** $\underline{Z}$ *is given by (5)* $\forall n \leq \hat{\hat{n}}$ *and otherwise it satisfies*

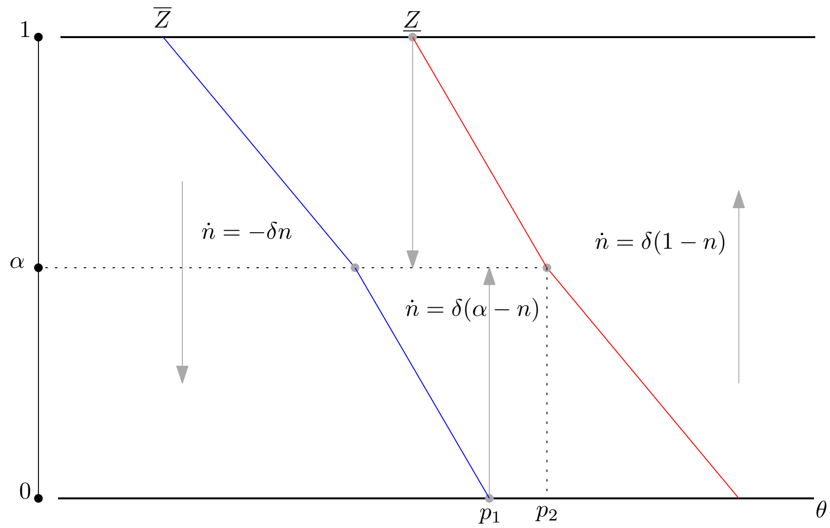
$$\frac{(1-n)}{1-\alpha} \int_{t=0}^{\infty} e^{-(\rho+\delta)t} \left[\overline{Z} + \overline{\varepsilon} + \gamma \left(1 - (1-n)e^{-\delta t} \right) \right] dt + \frac{(n-\alpha)}{1-\alpha} \left\{ \int_{t=0}^{\underline{t}} e^{-(\rho+\delta)t} \left[\overline{Z} + \overline{\varepsilon} + \gamma \left(\alpha + (n-\alpha)e^{-\delta t} \right) \right] - \int_{t=\underline{t}}^{\infty} e^{-(\rho+\delta)t} \left(\overline{\varepsilon} - \underline{\varepsilon} \right) dt \right\} = 0, \tag{6}$$

where $\underline{t} = -\frac{1}{\delta} \ln \frac{-\alpha - (\underline{Z}+\overline{\varepsilon})(\rho+2\delta)/\gamma(\rho+2\delta)}{n_0-\alpha}$ *and* $\hat{\hat{n}} = \alpha + \frac{(\rho+2\delta)(\overline{\varepsilon}-\underline{\varepsilon})-\gamma\delta}{\gamma\rho}$.

Proof. See Appendix. □

Figure (7) depicts the equilibrium in the case that $\overline{\varepsilon} - \underline{\varepsilon} \geq \frac{\gamma(\delta+\rho\alpha)}{\rho+2\delta}$ (graphically, it means that $p_1 < p_2$) and $\overline{\varepsilon} - \underline{\varepsilon} \geq \frac{\gamma(\delta+\rho(1-\alpha))}{\rho+2\delta}$. That is, the case in which if the economy starts at one threshold, it will never cross the other one, so that the equilibrium is given by equations (3) and (5).

Figure 7: Proposition 2, case $\overline{\varepsilon} - \underline{\varepsilon} \geq \frac{\gamma\delta+\gamma\rho\max\{\alpha,1-\alpha\}}{\rho+2\delta}$

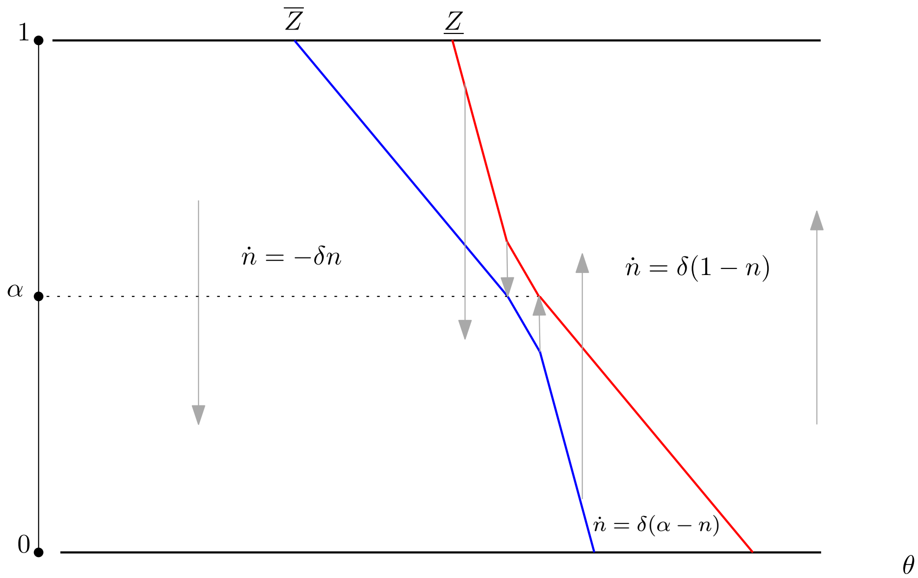


Consider a type-$\overline{\varepsilon}$ agent at some point on her threshold. Anywhere above α she knows that, with probability one, n will fall until it reaches zero. Small shocks leading θ either slightly to the left or to the right will both drive n downwards. Therefore, this part of the equilibrium threshold coincides with the upper dominance region for high-type agents. Now, consider that such agent is at some point on her threshold anywhere below α. Although a tiny shock moving θ to the left will lead to a drop in n, a shock to the right will make n bifurcates up until it reaches α. Thus, for a given decrease in n, the increase in θ that such agent needs to keep her indifferent between both networks is smaller, since her belief over n is a bit more optimistic. That is why the slope of the threshold is different below and above α.

An analogous reasoning applies for a type-$\underline{\varepsilon}$ agent that is called upon revising her choice at any point on her threshold. Below α, the only belief she can hold on the limiting case of vanishing shocks is that n will grow with probability one. Then, this part of the threshold coincides with the lower dominance region ($\underline{O}$). Above α, her belief is a little more pessimistic, since there is the possibility of n bifurcating down towards α.

The next figure depicts the equilibrium when there is less, but still a large amount of heterogeneity. There is a range of n (n is sufficient low) such that a type-$\overline{\varepsilon}$ agent on her threshold knows that, if the system bifurcates up, it will cross the other type's threshold at some point, and thereafter n will grow at a higher rate. Then, the quality an agent demands to be indifferent between the two networks is a little lower, i.e., the threshold is steeper. The same reasoning applies for a type-$\underline{\varepsilon}$ agent on her threshold. For n sufficiently large, she knows that if the system bifurcates down, n will eventually fall enough to cross the higher type's threshold, and thereafter the size of network B will decrease at a higher rate until it reaches zero.

Figure 8: Proposition 2 - $\frac{\gamma\delta}{\rho+2\delta} < \overline{\varepsilon} - \underline{\varepsilon} < \frac{\gamma\delta+\gamma\rho\min\{\alpha,1-\alpha\}}{\rho+2\delta}$



Not so large heterogeneity Now, lets analyze the case in which the condition in (2) does not hold. In terms of dominance regions, we have that $\overline{P} \geq \underline{Q}$, that is, agents who prefer B the most and are pessimistic about the size of that network demand a higher θ to be indifferent between A and B than agents who prefer A the most and are optimistic about the size of network B. In other words, the intersection between the 'non-dominant regions' of the two types is not empty, so there exist beliefs that would make them playing according to the same threshold. Surprisingly, whenever that is the case, there is a range of n such that different types of agents play exactly the same strategy.

Proposition 3. *Suppose $\overline{\varepsilon} - \underline{\varepsilon} \leq \frac{\gamma\delta}{\rho+2\delta}$. For all n between $n_1$and n_2, the two types of agents play according to the same (downward sloping) threshold. For all $n \leq n_1$,*

$$\underline{Z} = -\underline{\varepsilon} - \frac{\gamma\delta}{\rho+2\delta} - \frac{\gamma(\rho+\delta)}{\rho+2\delta}n$$

and $\overline{Z}$ satisfies (4). For all $n \geq n_2$,

$$\overline{Z} = -\overline{\varepsilon} - \frac{\gamma(\rho+\delta)}{\rho+2\delta}n$$

and $\underline{Z}$ satisfies (6). The bounds n_1 and n_2 are given by

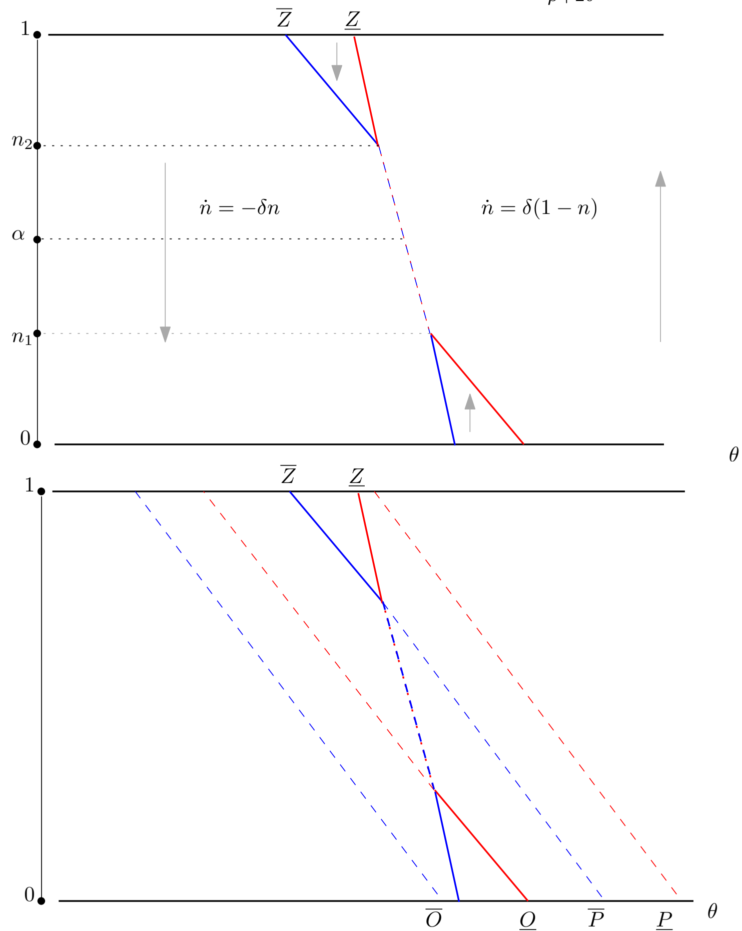
$$n_1 = \alpha \frac{(\overline{\varepsilon} - \underline{\varepsilon})(\rho + 2\delta)}{\gamma\delta},$$

$$n_2 = 1 - (1 - \alpha)\frac{(\overline{\varepsilon} - \underline{\varepsilon})(\rho + 2\delta)}{\gamma\delta}.$$

Proof. See Appendix. □

The equilibrium in this case is as depicted in figure 9 below.

Figure 9: Proposition 2 - $\overline{\varepsilon} - \underline{\varepsilon} < \frac{\gamma\delta}{\rho+2\delta}$



This result can be explained with the aid of Figure 7. If the agents' thresholds do not intersect each other, whenever $n > \alpha$, the type-$\overline{\varepsilon}$ agent knows everyone will be choosing A in the future, and whenever $n < \alpha$, the type-$\underline{\varepsilon}$ knows everyone will be choosing B in the future. Hence their equilibrium threshold coincide with the dominance bound that is closer to the other type's threshold. However, in case dominant regions intersect each other, if agents were to choose according to different threhsolds, then when n is in a neighborhood of α, the threshold $\underline{Z}(n)$ would be to the left of $\overline{Z}(n)$. But that cannot happen – and would anyway generate a different set of beliefs.

Therefore, the intuition for conformity is related to the intuition for why some agents have extreme beliefs whenever each type is playing differently. When n is small, that comes from the movement of n being determined by the choice of type-$\overline{\varepsilon}$ agents: they will choose B and since n is small, that is enough to ensure n will go up regardless of what type-$\underline{\varepsilon}$ agents do. So n immediately goes up, which leads type-$\underline{\varepsilon}$ agents to choose B as well. That implies type-$\underline{\varepsilon}$ agents know that when n is small, at the threshold where they are indifferent, n goes up with probability one. Hence their threshold coincides with the dominance bound closer to type-$\overline{\varepsilon}$ agents' threshold.

Intuitively, the existence of type-$\overline{\varepsilon}$ agents increase incentives for type-$\underline{\varepsilon}$ to choose B, while the existence of type-$\underline{\varepsilon}$ agents reduces incentives for type-$\underline{\varepsilon}$ to choose A, which makes them behave in a more similar way. That is particularly true when n is in an intermediate range so that the path of the economy will be decided by the actions of both groups.

4 The planner's problem with two-sided externalities

Agents are facing the choice between two networks, A and B. Suppose the flow utility agent i derives from being at network B is given by $u_i^B(\theta_t^B, n_t) = \theta_t^B + \nu^B n_t + \varepsilon_i^B$, and the flow utility from being at A is given by $u_i^A(\theta_t^A, n_t) = \theta_t^A + \nu^A(1 - n_t) + \varepsilon_i^A$. Hence n_t is the mass of agents currently on network B, $\nu^j > 0$ is a parameter measuring the relative importance of the network effect in network j (the importance of strategic complementarities), θ_t^j represents the quality of network j at time t, and ε_i^j captures an idiosyncratic preference for network j, $j \in \{A, B\}$.[9]

4.1 The case with ex-ante identical agents

In the case with ex-ante identical agents, $\varepsilon_i^j = 0$ for $j \in \{A, B\}$. Hence the relative payoff function can be written as

$$\pi_i(\theta_t, n_t) = \theta_t + \gamma n_t \; \forall i,$$

where $\theta \equiv \theta_B - \theta_A - \nu^A$ and $\gamma \equiv \nu^A + \nu^B$.

The planner's problem at time zero is to maximize

$$\mathcal{W} = \mathbb{E} \int_{t=0}^{\infty} e^{-\rho t} \left[n_t u_t^B + (1 - n_t) u_t^A \right] dt.$$

[9] If either ν^A or ν^B were equal to zero, we would have one sided externalities. The results here also apply for this case.

After a bit of algebra, one can see that this is equivalent to

$$\max \mathbb{E} \int_{t=0}^{\infty} e^{-\rho t} \left[n_t \left(\theta_t - \nu^A \right) + \gamma n_t^2 \right] dt.$$

If the planner increases n_0 in dn_0, this increase depreciates at a rate δ, so the effect in n_t is given by $dn_t = dn_0 e^{-\delta t}$. The optimality condition for the planner is

$$\mathbb{E} \int_0^{\infty} \frac{\partial \left[e^{-\rho t} \left(n_t(\theta_t - \nu^A) + \gamma n_t^2 \right) \right]}{\partial n_t} \frac{\partial n_t}{\partial n_0} dt = 0,$$

that is,

$$\mathbb{E} \int_{t=0}^{\infty} e^{(\rho+\delta)t} \left[\theta_t - \nu^A + 2\gamma n_t \right] dt = 0 \tag{7}$$

In order satisfy (7), the planner would make agents play according to a downward sloping threshold. Notice that this is very similar to the agent's problem: the agent is indifferent between A and B when $\mathbb{E} \int_{t=0}^{\infty} e^{(\rho+\delta)t} \left[\theta + \gamma n_t \right] dt = 0$. Proposition 2 relates the planner's solution to the decentralized solution.

Proposition 4. *Suppose there is a single type of agent in the economy. The decentralized equilibrium prescribes playing B whenever $\theta_t > Z^*(n_t)$ and A otherwise, where Z^* is given by*

$$Z^* = -\frac{\gamma\delta}{\rho + 2\delta} - \frac{\gamma\rho}{\rho + 2\delta} n. \tag{8}$$

The planner's solution prescribes playing B whenever $\theta_t > Z^P(n_t)$ and A otherwise, where Z^P is given by

$$Z^P = \nu^A - \frac{2\gamma\delta}{\rho + 2\delta} - \frac{2\gamma\rho}{\rho + 2\delta} n.$$

In the case of symmetric network effects, that is, $\nu^A = \nu^B$, the planner's solution becomes

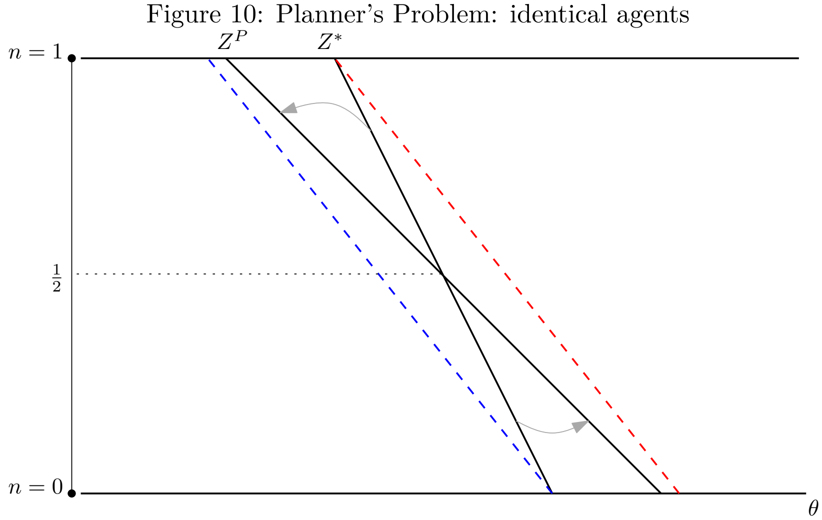
$$Z^P = -\frac{\gamma\delta}{(\rho + 2\delta)} + \frac{\gamma\rho}{2(\rho + 2\delta)} - \frac{2\gamma\rho}{\rho + 2\delta} n. \tag{9}$$

Proof. See Appendix. □

Notice that the planner gives twice as much weight to the current size of the networks than the agent does. Our intuition could lead us to think that the planner would push the agents towards the higher quality network, but it is not what happens here. The planner plays more against the fundamentals than the agents, since it takes into account the externality agents fail to internalize.

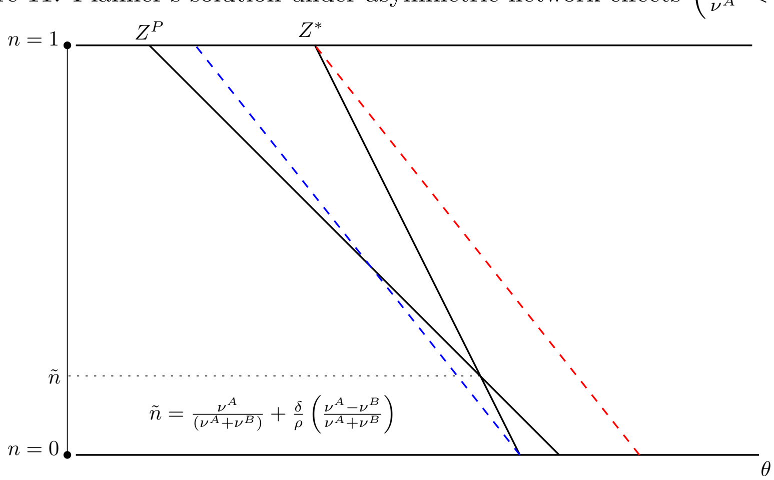
Figure (10) depicts the results in Proposition 4 when $\nu^A = \nu^B$. The planner rotates the threshold, so that the slope of its threshold is half the slope of the equilibrium one.

Figure 10: Planner's Problem: identical agents



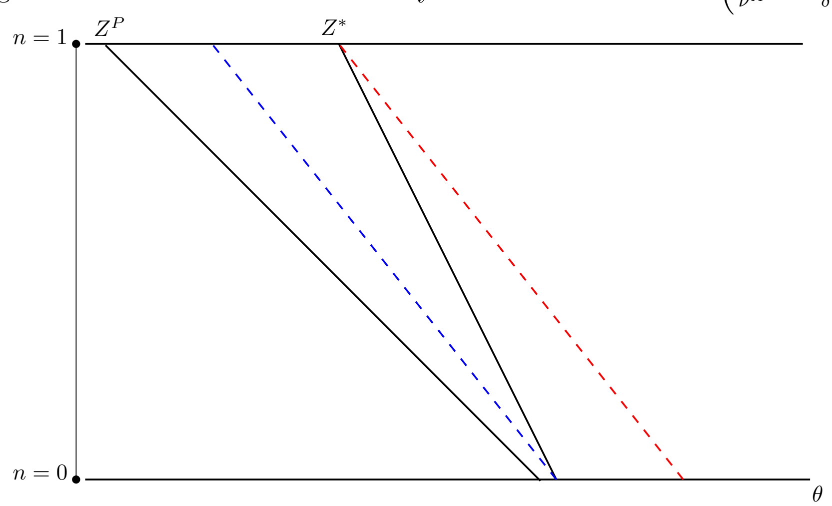
When the network effect is asymmetric, that is, $\nu^A \neq \nu^B$, the planner not only rotate the threshold around $n = 0.5$, but it also shifts the threshold in order to enlarge the region in which agents choose the network that generates more externalities. Figures (11) and (12) depicts the planner's solution when $\nu^B > \nu^A$.

Figure 11: Planner's solution under asymmetric network effects $\left(\frac{\nu^B}{\nu^A} < \frac{\rho+\delta}{\delta}\right)$



Notice that when the externality in one network is large enough in comparison to the other, the planner prescribes that a strictly dominated strategy must be played (the planner's threshold in figure (12) lies completely on the agents' lower dominance region). There is no belief an agent could hold at that region that would make her play B. However, the planner prescribes doing so. To see why, consider for example the case in which n

Figure 12: Planner's solution under asymmetric network effects $\left(\frac{\nu^B}{\nu^A} > \frac{\rho+\delta}{\delta}\right)$



is large. The planner takes into account that a lot of agents are stuck in network B (due to the timing frictions) and would benefit from the network effects generated by an additional increase in n.

4.2 The case with two types of agents

Proposition 4 shows that differences between ν^A and ν^B only add a constant to the planner's threshold. In this section, we extend the model for two types of agents and focus on the case in which $\nu^A = \nu^B = \nu$, for simplicity. The planner's problem in this case can be written as

$$\max \mathbb{E}\alpha \int_0^\infty e^{-\rho t} \left\{ \overline{n_t} \left[\theta^B + \nu n_t + \overline{\varepsilon}^B\right] + (1 - \overline{n_t}) \left[\theta^A + \nu(1 - n_t) + \overline{\varepsilon}^A\right] \right\} dt$$
$$+ (1-\alpha) \int_0^\infty e^{-\rho t} \left\{ \underline{n}_t \left[\theta^B + \nu n_t + \underline{\varepsilon}^B\right] + (1 - \underline{n}_t) \left[\theta^A + \nu(1 - n_t) + \underline{\varepsilon}^A\right] \right\} dt,$$

which is equivalent to

$$\max \mathbb{E} \int_0^\infty e^{-\rho t} \left\{ n_t \left[\theta - \frac{\gamma}{2} + \gamma n_t\right)\right] + \alpha \overline{n_t \varepsilon} + (1-\alpha) \underline{n_t \varepsilon} \right\} dt$$

Following the same reasoning as in the case of identical agents, we find that the optimality conditions for the planner are given by

(i)

$$\mathbb{E} \int_0^\infty e^{-(\rho+\delta)t} \left(\theta + \overline{\varepsilon} - \frac{\gamma}{2} + 2\gamma n_t\right) dt = 0$$

(ii)
$$\mathbb{E}\int_0^\infty e^{-(\rho+\delta)t}\left(\theta+\underline{\varepsilon}-\frac{\gamma}{2}+2\gamma n_t\right)dt=0$$

The next proposition characterizes the planner's solution.

Proposition 5. *Consider the model with two types of agents and linear payoff functions. The planner's solution is characterized by thresholds $\overline{Z}^P$ and $\underline{Z}^P$ as follows:*
(i) High-type planner:

If $\overline{\varepsilon}-\underline{\varepsilon}>\frac{2\gamma(\delta+\rho\alpha)}{\rho+2\delta}$:

$$\overline{Z}^P=\begin{cases}-\overline{\varepsilon}+\frac{\gamma}{2}-\frac{2\gamma(\rho+\delta)}{\rho+2\delta}n & \text{if } n\geq\alpha\\ -\overline{\varepsilon}+\frac{\gamma}{2}-\frac{2\alpha\gamma\delta}{\rho+2\delta}-\frac{2\gamma\rho}{\rho+2\delta}n & \text{if } n<\alpha\end{cases}. \tag{10}$$

If $\frac{2\gamma\delta}{\rho+2\delta}<\overline{\varepsilon}-\underline{\varepsilon}\leq\frac{2\gamma(\delta+\alpha\rho)}{\rho+2\delta}$*:*

For all $n\geq\hat{n}_p\equiv\alpha-\frac{(\overline{\varepsilon}-\underline{\varepsilon})(\rho+2\delta)-2\gamma\delta}{2\gamma\rho}$, $\overline{Z}^P$ *is given by equation (10), and otherwise it satisfies*

$$\frac{\alpha-n}{\alpha}\left\{\int_0^{\overline{t}_P}e^{-(\rho+\delta)t}\left[\overline{Z}^P-\frac{\gamma}{2}+\overline{\varepsilon}+2\gamma\left(\alpha-(\alpha-n)e^{-\delta t}\right)\right]dt+\int_{\overline{t}_P}^\infty e^{-(\rho+\delta)t}\left(\overline{\varepsilon}-\underline{\varepsilon}\right)dt\right\}$$
$$+\frac{n}{\alpha}\int_0^\infty e^{-(\rho+\delta)t}\left[\overline{Z}^P-\frac{\gamma}{2}+\overline{\varepsilon}+2\gamma ne^{-\delta t}\right]dt=0,$$

where $\overline{t}_P=\frac{1}{\delta}\ln\frac{\alpha+[\left(\overline{Z}^P-\gamma/2+\underline{\varepsilon}\right)(\rho+2\delta)+2\gamma\delta]/2\gamma(\rho+\delta)}{\alpha-n}$.

If $\overline{\varepsilon}-\underline{\varepsilon}\leq\frac{2\gamma\delta}{\rho+2\delta}$*:*

$\forall n\leq n_1^P\equiv\alpha\frac{(\overline{\varepsilon}-\underline{\varepsilon})(\rho+2\delta)}{2\gamma\delta}$, $\overline{Z}$ *satisfies (10) and* $\forall n\geq n_2^P\equiv 1-(1-\alpha)\frac{(\overline{\varepsilon}-\underline{\varepsilon})(\rho+2\delta)}{\gamma\delta}$, $\overline{Z}^P=-\overline{\varepsilon}+\frac{\gamma}{2}-\frac{2\gamma(\rho+\delta)}{\rho+2\delta}n$.

(ii) Low-type planner:

If $\overline{\varepsilon}-\underline{\varepsilon}>\frac{2\gamma(\delta+\rho(1-\alpha))}{\rho+2\delta}$,

$$\underline{Z}^P=\begin{cases}-\underline{\varepsilon}+\frac{\gamma}{2}-\frac{2\gamma\delta(1+\alpha)}{\rho+2\delta}-\frac{2\gamma\rho}{\rho+2\delta}n & \text{if } n>\alpha\\ -\underline{\varepsilon}+\frac{\gamma}{2}-\frac{2\gamma\delta}{\rho+2\delta}-\frac{2\gamma(\rho+\delta)}{\rho+2\delta}n & \text{if } n\leq\alpha\end{cases}. \tag{11}$$

If $\frac{2\gamma\delta}{\rho+2\delta}<\overline{\varepsilon}-\underline{\varepsilon}\leq\frac{2\gamma(\delta+\rho(1-\alpha))}{\rho+2\delta}$:

For all $n\leq\hat{\hat{n}}_p=\alpha+\frac{(\overline{\varepsilon}-\underline{\varepsilon})(\rho+2\delta)-2\gamma\delta}{2\gamma\rho}$, $\underline{Z}^P$ *is given by equation (11), and otherwise it satisfies*

$$\frac{(1-n)}{1-\alpha}\int_{t=0}^{\infty} e^{-(\rho+\delta)t}\left[\underline{\theta}^P - \frac{\gamma}{2} + \underline{\varepsilon} + 2\gamma\left(1-(1-n)e^{-\delta t}\right)\right]dt$$
$$+\frac{(n-\alpha)}{1-\alpha}\left\{\int_{t=0}^{\underline{t}_P} e^{-(\rho+\delta)t}\left[\underline{\theta}^P - \frac{\gamma}{2} + \underline{\varepsilon} + 2\gamma\left(\alpha+(n-\alpha)e^{-\delta t}\right)\right] - \int_{t=\underline{t}_P}^{\infty} e^{-(\rho+\delta)t}\left(\overline{\varepsilon}-\underline{\varepsilon}\right)dt\right\} = 0,$$

where $\underline{t}_P = -\frac{1}{\delta}\ln\frac{-\left(\underline{\theta}^P - \frac{\gamma}{2}+\underline{\varepsilon}\right)(\rho+2\delta)/2\gamma(\rho+2\delta)-\alpha}{n-\alpha}$.

If $\overline{\varepsilon}-\underline{\varepsilon} \leq \frac{2\gamma\delta}{\rho+2\delta}$*:*

$\forall n \leq n_1^P \equiv \alpha\frac{(\overline{\varepsilon}-\underline{\varepsilon})(\rho+2\delta)}{2\gamma\delta}$, $\underline{Z}^P = -\overline{\varepsilon} + \frac{\gamma}{2} - \frac{2\gamma\delta}{\rho+2\delta} - \frac{2\gamma(\rho+\delta)}{\rho+2\delta}n$, *and* $\forall n \geq n_2^P \equiv 1-(1-\alpha)\frac{(\overline{\varepsilon}-\underline{\varepsilon})(\rho+2\delta)}{\gamma\delta}$, $\underline{Z}^P$ *satisfies (11).*

Proof. See Appendix. □

The planner's solution has some interesting properties. The threshold is always flatter than the agents', meaning that the planner sacrifices quality in order to explore strategic complementarities. Moreover, the region in which the planner prescribes that the same strategy must be played by different types is always larger, showing that the planner cares less about idiosyncratic preferences. In case the equilibrium threshold of both types coincide for some values of n, the range of values for n where agents choose different actions is twice as big as the analogous region for the planner. Figures (13) to (15) depicts the planner's solutions for several ranges of heterogeneity in comparison to the decentralized solution.

5 Final remarks

TBW

A Proofs

A.1 Proof of Proposition 1

TBW

A.2 Proof of Proposition 2

Suppose that $\mu, \sigma \to 0$ and that $\overline{\varepsilon}-\underline{\varepsilon} > \frac{\gamma\delta}{\rho+2\delta}$. The following lemma, based on Theorem 2 in Burdzy et al. (1998), helps us compute the equilibrium.

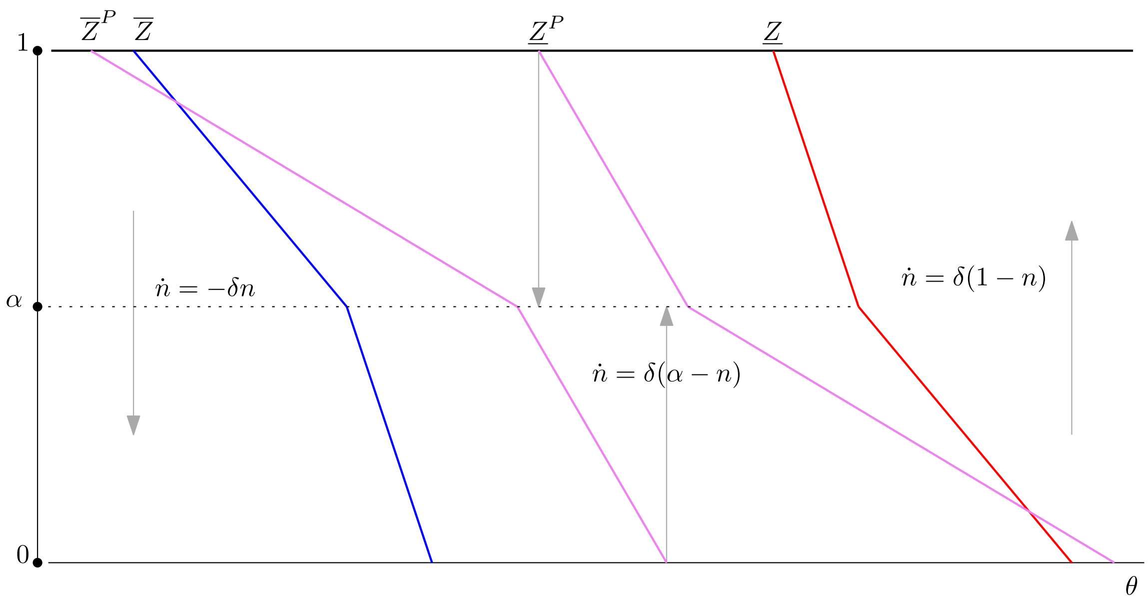


Figure 13: Planner's solution when $\overline{\varepsilon} - \underline{\varepsilon} > \frac{2\gamma(\delta+\rho\max\{\alpha,(1-\alpha)\})}{\rho+2\delta}$

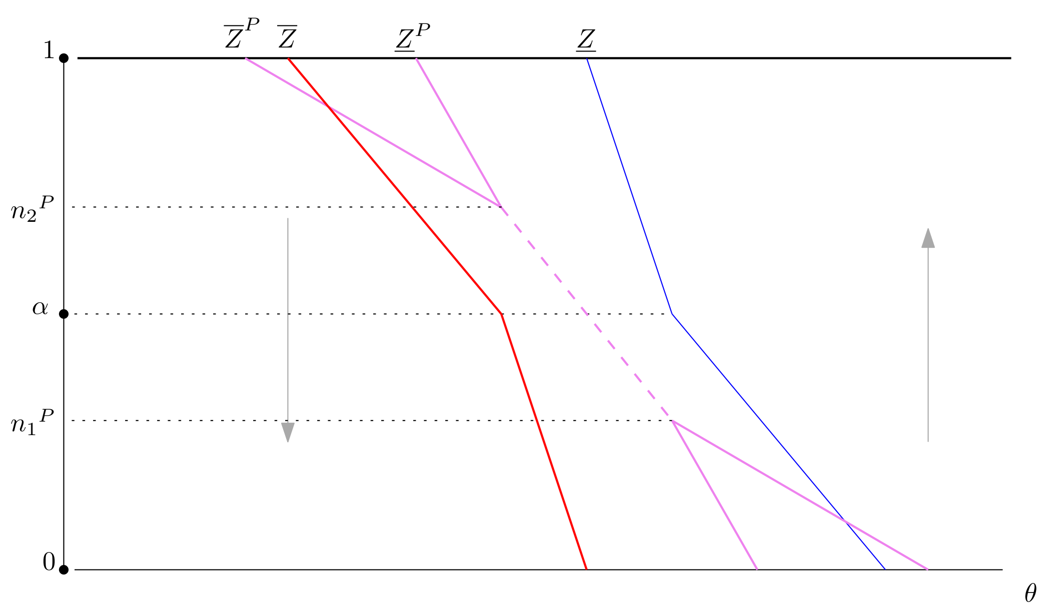


Figure 14: Planner's solution when $\frac{2\gamma\delta}{\rho+2\delta} < \overline{\varepsilon} - \underline{\varepsilon} \leq \frac{2\gamma(\delta+\rho\max\{\alpha,(1-\alpha)\})}{\rho+2\delta}$

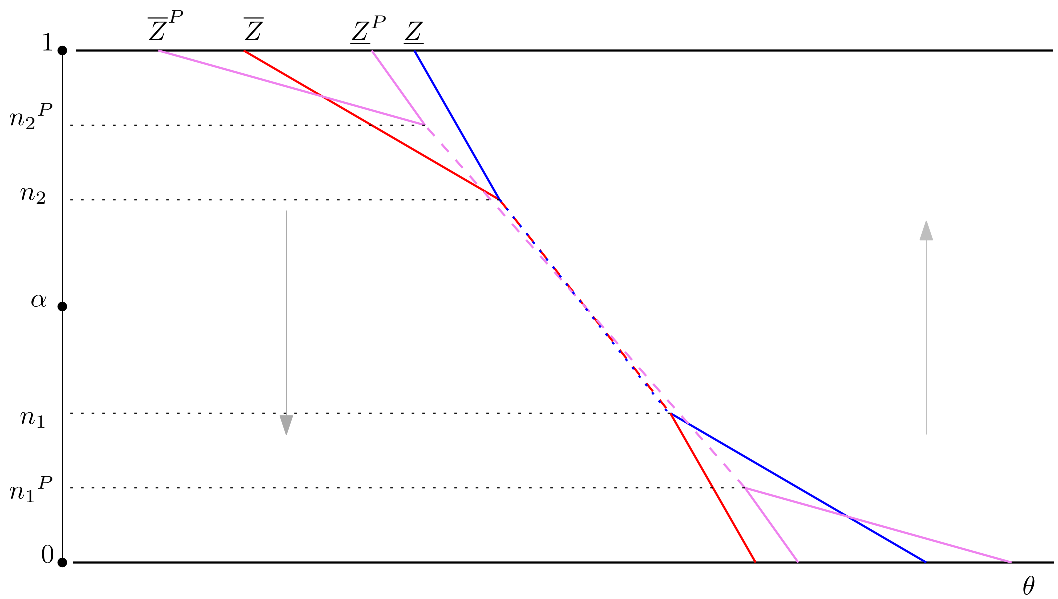


Figure 15: Planner's solution when $\overline{\varepsilon} - \underline{\varepsilon} \leq \frac{2\gamma\delta}{\rho+2\delta}$

Lemma 2. *Suppose each type of agent plays according to a distinct threshold, that is,* $\overline{Z}(n) < \underline{Z}(n) \, \forall n$. *Consider a point* (θ, n) *with* $\theta = Z_i^*(n)$. *As* $\mu, \sigma \to 0$, *the time it takes for the system to bifurcate either up or down converges to zero. Moreover, the probabilities of as upward or a downward bifurcation are computed as follows:*

(i) Consider a point (θ, n) *with* $\theta = \overline{Z}(n)$.

$$P(up) = \begin{cases} 0 & \text{if } n \geq \alpha \\ 1 - \frac{n}{\alpha} & \text{if } n < \alpha \end{cases}$$

and $P(down) = 1 - P(up)$.

(ii) Consider a point (θ, n) *eith* $\theta = \underline{Z}(n)$.

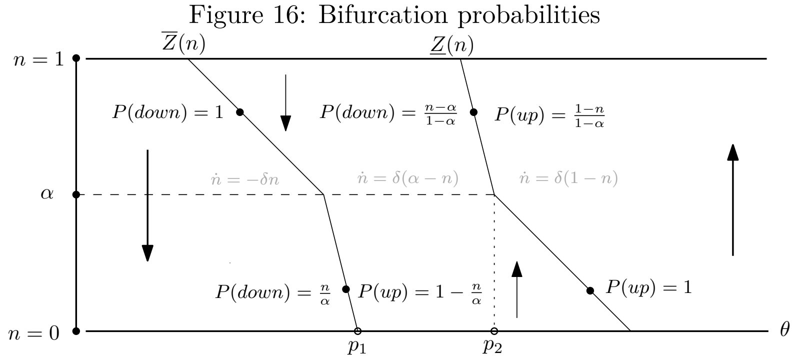
$$P(up) = \begin{cases} \frac{1-n}{1-\alpha} & \text{if } n \geq \alpha \\ 1 & \text{if } n < \alpha \end{cases}$$

and $P(down) = 1 - P(up)$.

Proof. The proof follows imediatelly from Burdzy et al. (1998). The dynamics around $\overline{Z}$ is given by

$$\dot{n}_t = \begin{cases} \delta(\alpha - n_t) & \text{if } \theta_t > \overline{Z}(n_t) \\ -\delta n_t & \text{if } \theta_t < \overline{Z}(n_t) \end{cases},$$

Figure 16: Bifurcation probabilities



which can be rewritten as

$$\dot{n}_t = \alpha \dot{x}_t = \begin{cases} \delta\alpha(1 - x_t) & \text{if } \theta_t > \overline{Z}(n_t) \\ -\delta\alpha x_t & \text{if } \theta_t < \overline{Z}(n_t) \end{cases},$$

where $x_t = n_t/\alpha$, in order to apply Theorem 2 in Burdzy et al. (1998) directly. The dynamics around $\underline{Z}$ can also be rewritten in order to fit in that Theorem. $\square$

Figure (16) below shows the bifurcation probabilities along the thresholds. The intuition is that the probability of n going up or down depends on the rate at which it goes each direction (in case it does), and once it has headed off in one direction, it does not revert to Z_i^*. Using these bifurcation probabilities, we can compute the equilibrium by equating the expected payoff of agents to zero on their thresholds.

Type-$\overline{\varepsilon}$ threshold: For now, lets assume that the equilibrium is such that the distance between the thresholds of the two types of agents is big enough so that, in the limit as $\mu, \sigma \to 0$, if the economy starts at some point on $\overline{Z}$, it will never cross $\underline{Z}$. Geometrically, it is equivalent to assuming that p_1 is located to the left of p_2 in figure (16). We will show that it is the case whenever $\overline{\varepsilon} - \underline{\varepsilon} > \frac{\gamma(\delta+\rho\alpha)}{\rho+2\delta}$.

We know that, on the equilibrium threshold, each type of agent is indifferent between the two networks if they expect others to play according to their thresholds. Consider an agent $i \in [0, \alpha]$ at some point (θ_0, n_0) with $\theta_0 = \overline{Z}(n_0)$, i.e., at some point on her threshold. If $n \geq \alpha$, we know by Lemma 2 that she expects n to decrease with probability one. Thus, the size of network B that such agent expects for future dates is given by $n_t = n_0 e^{-\delta t}$. That is, she expects that only the ones that were initially at network B and did not get the chance to leave it until time t will still be at that network. If $n_0 < \alpha$, with some probability n will bifurcate down towars zero, but with some probability it will bifurcate up towars α, in which case the expect n for future dates will be $n_t = \alpha - (\alpha - n_0)e^{-\delta t}$.

We proceed by dividing the computation of each threshold in two parts, since the beliefs agents hold are different above and below $n_0 = \alpha$.

Solving the equation below for $\overline{Z}$ gives us the first line of equation (3), the case in which $n_0 \geq \alpha$.

$$\int_{t=0}^{\infty} e^{-(\rho+\delta)t} \left[\overline{Z} + \overline{\varepsilon} + \gamma n_0 e^{-\delta t}\right] dt = 0$$

This part of agent i's threshold, $i \in [0, \alpha]$, coincides with the upper dominance region, $\overline{P}$, in which such agents hold the most pessismistic belief (everyone who gets the chance to choose a network will play B). The second part of equation (3), for the case $n_0 < \alpha$, is obtained by solving the following equation for $\overline{Z}$:

$$\underbrace{\frac{\alpha - n_0}{\alpha}}_{P(\text{up})} \int_{t=0}^{\infty} e^{-(\rho+\delta)t} \left[\overline{Z} + \overline{\varepsilon} + \gamma \left(\alpha - (\alpha - n_0)e^{-\delta t}\right)\right] dt + \underbrace{\frac{n_0}{\alpha}}_{P(\text{down})} \int_{t=0}^{\infty} e^{-(\rho+\delta)t} \left[\overline{Z} + \overline{\varepsilon} + \gamma n_0 e^{-\delta t}\right] = 0.$$

The first term of the sum is the probability of an upward bifurcation times the discounted payoff when the agent expects n_t to grow until it approaches α. The second one is the probability of a downward bifurcation times the discounted payoff when the agent expects n_t to decrease towards zero.

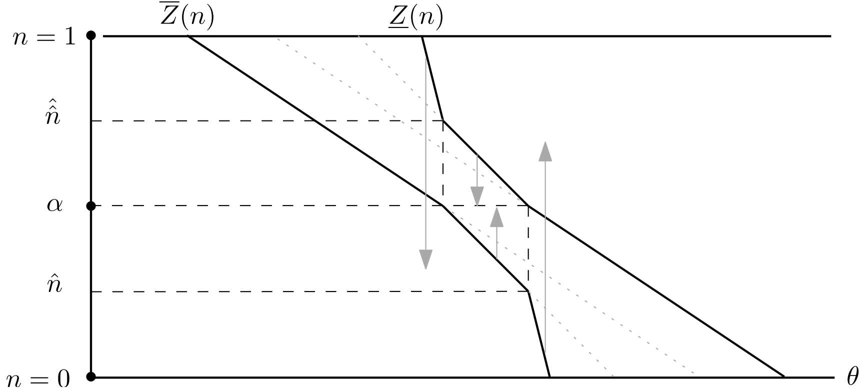
We still need to check the condition under which the path of a system starting at any point (θ_0, n_0) with $\theta_0 = \overline{Z}(n_0)$ does not ever cross the other threshold, $\underline{Z}$. It suffices to find a condition that guarantees that $\overline{Z}(0) < \underline{Z}(\alpha)$, that is,

$$-\overline{\varepsilon} - \frac{\alpha\gamma\delta}{\rho + 2\delta} < -\underline{\varepsilon} - \frac{\gamma\delta}{\rho + 2\delta} - \frac{\gamma\alpha(\rho + \delta)}{\rho + 2\delta}$$

$$\iff \overline{\varepsilon} - \underline{\varepsilon} > \frac{\gamma(\delta + \rho\alpha)}{\rho + 2\delta}.$$

Now, assume that the condition above does not hold. Instead, we have that $\frac{\gamma(\delta+\rho\alpha)}{\rho+2\delta} \geq \overline{\varepsilon} - \underline{\varepsilon} > \frac{\gamma\delta}{\rho+2\delta}$. In that case, an individual making a choice at some point on her threshold needs to take into account the possibility that, depending on the initial state (θ_0, n_0), the system may bifurcate up but not only towards $n_t = \alpha$. The size of network B may increase at a lower rate until it crosses the other agents' threshold, and then continue to grow towards $n = 1$. The figure below exemplifies this case.

Figure 17: Intermediate range of heterogeneity



Define $\hat{n} \equiv \overline{Z}^{-1}\left(\underline{Z}(\alpha)\right) = \alpha - \frac{(\rho+2\delta)(\overline{\varepsilon}-\underline{\varepsilon})-\gamma\delta}{\gamma\rho}$. For all $n_0 > \hat{n}$, the threshold $\overline{Z}$ is given by equation (3). But when $n_0 \leq \hat{n}$, the agent takes into account that, if the system bifurcates up, n_t will grow at a rate $\delta(\alpha - n_t)$ until it reaches $\hat{n}$, and then it will grow at a higher rate, $\delta(1-n_t)$, since the system will have crossed the threshold of all other agents in the economy. Thus, the equation expressing indifference between networks A and B of an agent $i \in [0,\alpha]$ choosing at a point (θ_0, n_0) with $\theta_0 = \overline{Z}(n_0)$, $n_0 \leq \hat{n}$, is

$$\underbrace{\frac{(\alpha-n_0)}{\alpha}}_{P(\text{up})}\left\{\int_{t=0}^{\bar{t}} e^{-(\rho+\delta)t}\left[\overline{Z}+\overline{\varepsilon}+\gamma\underbrace{\left(\alpha-(\alpha-n_0)e^{-\delta t}\right)}_{n_t \text{ growing at rate } \delta(\alpha-n)}\right]dt + \int_{t=\bar{t}}^{\infty} e^{-(\rho+\delta)t}\left[\overline{Z}+\overline{\varepsilon}+\gamma\underbrace{\left(1-(1-n_0)e^{-\delta t}\right)}_{n_t \text{ growing at rate } \delta(1-n)}\right.\right.$$

$$+\underbrace{\frac{n_0}{\alpha}}_{P(\text{down})}\int_{t=0}^{\infty}\left[\overline{Z}+\overline{\varepsilon}+\gamma\underbrace{n_0e^{-\delta t}}_{n_t \text{ falling}}\right]dt = 0,$$

where $\bar{t} = -\frac{1}{\delta}\ln\frac{\alpha-\underline{Z}^{-1}(\overline{Z}(n_0))}{\alpha-n_0} = -\frac{1}{\delta}\ln\frac{\alpha+[(\overline{Z}+\underline{\varepsilon})(\rho+2\delta)+\gamma\delta]/\gamma(\rho+2\delta)}{\alpha-n_0}$ is the time at which the system crosses$\underline{Z}$ in the case it bifurcates up.[10] Solving for $\overline{Z}$ gives us the last piece of the equilibrium threshold for type-$\overline{\varepsilon}$ agents. Notice that the second integral in the first line is equivalent to $\int_{t=\bar{t}}^{\infty} e^{-(\rho+\delta)t}\left(\overline{\varepsilon}-\underline{\varepsilon}\right)dt$, which is the difference between a type-$\overline{\varepsilon}$ agent's payoff and a type-$\underline{\varepsilon}$ agent's payoff at that point (the latter has zero payoff since at $\underline{t}$ the system is exactly on her threshold).

Type-$\underline{\varepsilon}$ threshold: The same reasoning applies for an agent $i \in (\alpha, 1]$. Consider that such agent receives a chance to revise her network choice at some point (θ_0, n_0) on her threshold, $\theta_0 = \underline{Z}(n_0)$. First, suppose that the distance between the thresholds is such

[10] The $\underline{Z}$ used to compute $\bar{t}$ is given by the second line of equation (5), since $\hat{n} < \alpha$.

that whenever the economy starts at some point on $\underline{Z}$, it will never cross $\overline{Z}$ (we will find the condition that guarantees that later). By Lemma 2, whenever $\alpha \leq n$, the system goes up with probability one. Whenever $\alpha > n$, the probabilities of bifurcating up (towards $n = 1$) or down (towards $n = \alpha$) are $(1-n)/(1-\alpha)$ and $(n-\alpha)/(1-\alpha)$, respectively. We can then compute the threshold. The first part of equation (5) is obtained by solving

$$\underbrace{\frac{(1-n_0)}{1-\alpha}}_{P(\text{up})}\int_{t=0}^{\infty} e^{-(\rho+\delta)t}\left[\underline{Z}+\underline{\varepsilon}+\gamma\left(1-(1-n_0)e^{-\delta t}\right)\right]dt+\underbrace{\frac{(n_0-\alpha)}{1-\alpha}}_{P(\text{down})}\int_{t=0}^{\infty} e^{-(\rho+\delta)t}\left[\underline{Z}+\underline{\varepsilon}+\gamma\left(\alpha+(n_0-\alpha)e^{-\delta t}\right.\right.$$

for $\underline{Z}$, and the second one, by solving

$$\int_{t=0}^{\infty} e^{-(\rho+\delta)t}\left[\underline{Z}+\underline{\varepsilon}+\gamma\left(1-(1-n_0)e^{-\delta t}\right)\right]dt = 0.$$

The last equation shows that this part of the equilibrium threshold coincides with the lower dominance region, $\underline{O}$, in which such agents expect all others to play B in the future. Now, lets check the condition under which a system starting at any point (θ_0, n_0) with $\theta_0 = \underline{Z}(n_0)$ will never reach the threshold $\overline{Z}$. It suffices to check whether $\overline{Z}(\alpha) < \underline{Z}(1)$, which is true whenever

$$-\overline{\varepsilon} - \frac{\gamma\alpha(\rho+\delta)}{\rho+2\delta} < -\underline{\varepsilon} - \frac{\gamma\delta(1+\alpha)}{\rho+2\delta} - \frac{\gamma\rho}{\rho+2\delta}$$

$$\iff \overline{\varepsilon} - \underline{\varepsilon} > \frac{\gamma\left[\delta+\rho(1-\alpha)\right]}{\rho+2\delta}.$$

Now, assume instead that $\frac{\gamma[\delta+\rho(1-\alpha)]}{\rho+2\delta} \geq \overline{\varepsilon} - \underline{\varepsilon} > \frac{\gamma\delta}{\rho+2\delta}$. Define $\hat{\hat{n}} \equiv \underline{Z}^{-1}\left(\overline{Z}(\alpha)\right) = \alpha + \frac{(\rho+2\delta)(\overline{\varepsilon}-\underline{\varepsilon})-\gamma\delta}{\gamma\rho}$. For all $n_0 < \hat{\hat{n}}$, the threshold of type-$\underline{\varepsilon}$

agents is still given by equation (5), but if $n_0 \geq \hat{\hat{n}}$, $\underline{Z}$ satisfies

$$\underbrace{\frac{(1-n_0)}{1-\alpha}}_{P(\text{up})}\int_{t=0}^{\infty} e^{-(\rho+\delta)t}\left[\overline{Z}+\overline{\varepsilon}+\gamma\underbrace{\left(1-(1-n_0)e^{-\delta t}\right)}_{n_t \text{ growing towards } 1}\right]dt$$

$$+\underbrace{\frac{(n_0-\alpha)}{1-\alpha}}_{P(\text{down})}\left\{\int_{t=0}^{\underline{t}} e^{-(\rho+\delta)t}\left[\overline{Z}+\overline{\varepsilon}+\gamma\underbrace{\left(\alpha+(n_0-\alpha)e^{-\delta t}\right)}_{n_t \text{ falling at rate } \delta(n-\alpha)}\right] + \int_{t=\underline{t}}^{\infty} e^{-(\rho+\delta)t}\left[\overline{Z}+\overline{\varepsilon}+\gamma\underbrace{\left(1-(1-n)e^{-\delta t}\right)}_{n_t \text{ falling at rate } \delta n}\right]dt\right\}$$

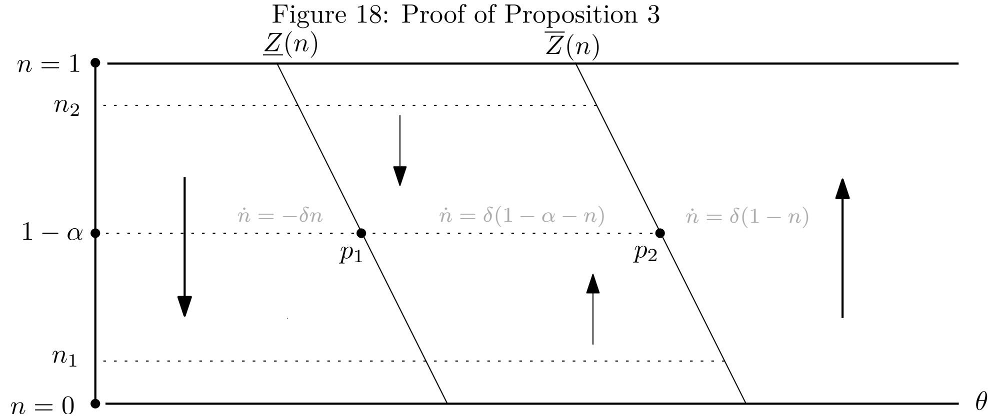
where $\underline{t} = -\frac{1}{\delta} \ln \frac{\overline{Z}^{-1}(\underline{Z}(n_0)) - \alpha}{n_0 - \alpha} = -\frac{1}{\delta} \ln \frac{-\alpha - (\underline{Z} + \overline{\varepsilon})(\rho + 2\delta)/\gamma(\rho + 2\delta)}{n_0 - \alpha}$. [11] The last term of the sum can be substituted by $-\int_{t=\underline{t}}^{\infty} e^{-(\rho+\delta)t} (\overline{\varepsilon} - \underline{\varepsilon})\, dt$, which is the difference between the payoffs of the two types of agents since at $\underline{t}$ the system is exactly on type-$\overline{\varepsilon}$'s threshold. □

A.3 Proof of Proposition 3

Suppose $\overline{\varepsilon} - \underline{\varepsilon} \leq \frac{\gamma\delta}{\rho + 2\delta}$ and that $\overline{Z}(n) \neq \underline{Z}(n)$ for all $n \in (n_1, n_2)$. Then, we have that, for each n, either (i) $\overline{Z}(n) > \underline{Z}(n)$ or (ii) $\overline{Z}(n) < \underline{Z}(n)$.

(i) To show that this cannot be the case, lets look at the dynamics of n implied by $\overline{Z}(n) > \underline{Z}(n)$ in figure (18). Notice that, for all $n \geq 1 - \alpha$, a type-$\underline{\varepsilon}$ agent at any point in $\underline{Z}$ expects n to fall with probability one, while a type-$\overline{\varepsilon}$ agent at $\overline{Z}$ expects n to grow with some probability. Besides having a higher idiosyncratic preference for network B, the type-$\overline{\varepsilon}$ agent at $\overline{Z}$ faces a higher θ and has a better belief about n than a type-$\underline{\varepsilon}$ agent at $\underline{Z}$. Then, they cannot both be indifferent between A and B. Likewise, for all $n < 1 - \alpha$, a type-$\overline{\varepsilon}$ agent on $\overline{Z}$ expects n to go up with probability one, while a type-$\underline{\varepsilon}$ agent at $\underline{Z}$ expects n to fall with some probability. Since the θ, the expected n and the parameter ε are higher for the type-$\overline{\varepsilon}$ agent, they cannot both be indifferent. Thus, $\nexists n$ such that $\overline{Z}(n) > \underline{Z}(n)$.

Figure 18: Proof of Proposition 3



(ii) Notice that if $\overline{Z}(\alpha) < \underline{Z}(\alpha)$, we have that $\underline{Z}(\alpha) = \underline{O}(\alpha)$ and $\overline{Z}(\alpha) = \overline{P}(\alpha)$. Then, $\underline{O}(\alpha) > \overline{P}(\alpha) \Rightarrow \overline{\varepsilon} - \underline{\varepsilon} > \frac{\gamma\delta}{\rho + 2\delta}$. Contradiction. Thus, at $n = \alpha$, all agents must play the same strategy. Moreover, given that $\overline{\varepsilon} - \underline{\varepsilon} \leq \frac{\gamma\delta}{\rho + 2\delta}$, we have that $\hat{n} = \alpha - \frac{(\rho + 2\delta)(\overline{\varepsilon} - \underline{\varepsilon}) - \gamma\delta}{\gamma\rho} \geq \alpha$ and $\hat{\hat{n}} = \alpha + \frac{(\rho + 2\delta)(\overline{\varepsilon} - \underline{\varepsilon}) - \gamma\delta}{\gamma\rho} \leq \alpha$. Thus, under the assumption that $\overline{Z}(n) < \underline{Z}(n)$, by Proposition 2 we have that, $\forall n < \alpha$, $\overline{Z}$ is given by equation (4) and $\underline{Z} = \underline{O} = -\underline{\varepsilon} - \frac{\gamma\delta}{\rho + 2\delta} - \frac{\gamma(\rho + \delta)}{\rho + 2\delta} n$. Also, $\forall n > \alpha$, $\underline{Z}$ is given by equation (6) and

[11] The $\overline{Z}$ used to compute $\underline{t}$ is given by the firs line of equation (3), since $\hat{n} > \alpha$.

$\overline{Z} = \overline{P} = -\overline{\varepsilon} - \frac{\gamma(\rho+\delta)}{\rho+2\delta}n$. However, one can verify that the two thresholds cross at $n = n_1$ and at $n = n_2$, and $\forall n \in (n_1, n_2)$ we have that $\overline{Z}(n) > \underline{Z}(n)$. Contradiction. The two types cannot play different strategies if $n \in (n_1, n_2)$.

$\square$

A.4 Proof of Proposition 4

The proof of Proposition 4 is equivalent to the proof of Proposition 2 if we substitute the flow playoff of the agent, $\pi(.)$, by ${Z_t}^P - \frac{\gamma}{2} + 2\gamma n_t$.

$\square$

A.5 Proof of Proposition 5

The proof of Proposition 5 is equivalent to the proof of Proposition 2 if we substitute the flow playoff of the agent, $\pi(.)$, by $\underline{Z_t}^P + \underline{\varepsilon} - \frac{\gamma}{2} + 2\gamma n_t$ and by $\overline{Z_t}^P + \overline{\varepsilon} - \frac{\gamma}{2} + 2\gamma n_t$. $\square$

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