Large Contests

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Abstract

We consider contests with many, possibly heterogeneous, players and prizes that include many existing contest models as special cases. We show that the equilibria of such contests are approximated by an appropriately defined set of incentive-compatible individually-rational single-agent mechanisms.

This approach makes it possible to approximate the equilibria of contests whose exact equilibrium characterization is complicated, as well as the equilibria of contests for which there is no existing equilibrium characterization. This facilitates contest design, welfare analysis, and comparative statics.

1 Introduction

There are many settings in which economic agents compete for prizes by expending resources. Some of these settings, such as rent-seeking scenarios, political campaigns, competitions for promotions, and some research and development races, typically involve a small number of competitors and prizes; other settings, such as college admissions, competitions by employers for workers on a national or international level (e.g., hospitals for residents, universities for

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faculty), sales competitions in large firms, and certain sports competitions (e.g., marathons) involve a large number of competitors and prizes.

The competitors often differ in their abilities, technologies, access to capital, and prior investments. They may also receive differential treatment in the contest, and have differing valuations for the prizes. The prizes are often heterogeneous - colleges differ in their prestige, jobs differ in their characteristics - which may further accentuate the asymmetries among competitors. Existing contest models, however, typically do not accommodate substantial asymmetries among competitors and heterogeneity in prizes, because each feature separately, and their combination in particular, present significant technical difficulties. This limits the study of real-world contests, in which asymmetric competitors and heterogeneous prizes are rather common.

The goal of this paper is to expand our understanding of contests with asymmetric competitors and heterogeneous prizes by approximating their equilibrium outcome when the number of players and prizes is large (but finite). We show that for a broad specification of contests, as the number of players and prizes increases, players' equilibrium bids and resulting allocation of prizes are approximated by a subset of the incentive compatible (IC) and individually rational (IR) mechanisms that allocate a limit set of prizes to a single agent whose possible types correspond to a limit set of players.

More precisely, we begin with two distributions, one of agent types and one of prizes, and a continuous utility function for the agent, whose arguments are an agent type, a prize, and a bid. The utility increases in the prize and decreases in the bid, but need not be quasi-linear in the bid or monotonic in the agent type. We then define a sequence of contests $1, 2, \ldots$, such that the n-th contest has n players and prizes (some of which may be worth 0). The players compete by each choosing a non-negative bid, simultaneously and independently. The player with the highest bid obtains the highest prize (prize n), etc. The j-th prize in the n-th contest is the prize that is the j-th n-quantile of the prize distribution. The Bernoulli utility of the i-th player in the n-th contest corresponds to the agent utility with an agent type that is the i-th n-quantile of the agent type distribution. This contest is a game of complete information, so the equilibria are typically in mixed strategies. The complete information assumption, which we discuss in in Section 1.1, can be interpreted as each player knowing the set of prizes and the empirical distribution of the players she is facing. The generality of the agent's utility function accommodates a wide range of asymmetries among players and heterogeneity in prizes. The approximating single-agent IC-IR mechanisms we consider allocate to each agent type a distribution over prize-bid pairs in a way that is consistent with an "inverse tariff" that assigns a prize to each bid.

The rough intuition for the approximation is as follows. In any contest, a player can bid 0 and secure the lowest prize, so IR holds. As the number of players increases, the competition they face becomes similar. That is, the mappings between bids and players' percentile rankings (given the other players' equilibrium strategies) become similar for all players, and coincide in the limit. This "almost" implies that also the mappings between bids and distributions of prizes that the players face become similar, and coincide in the limit. Therefore, in the limit, each player can mimic any other player in terms of what they obtain. Moreover, by the law of large numbers, the common limit mapping is deterministic, so that each bid maps to a single prize. This yields an inverse tariff such that in the limit players choose their bids as in a mechanism-design setting in which a single agent faces the inverse tariff. In particular, the mechanism defined by this mapping is IC.

This intuition is incomplete, however, since for some bids the distributions of prizes may not be similar for all players (despite the fact that the percentile rankings are), even when the number of players grows large. Therefore, a player's bid may not be sufficient for determining, even approximately, the prize that the player gets in a large contest. For example, if the limit percentile ranking of a bid t is $1/2$ and there are half as many identical prizes as players, then it is unclear whether in a large contest a player obtains a prize by bidding t. This is the case in the setting of Section 3.1, where by bidding slightly above t some players obtain a prize with a relatively high probability while other players obtain a prize with a relatively low probability, even when the number of players grows large. In addition, even in settings in which this problem does not arise the notion of approximation implied by the intuition is substantially weaker than the ones we are able to obtain.

Our most general, but weakest, result applies to agent utilities that depend arbitrarily (but continuously) on the agent's type. For the result, we consider for each player the equilibrium distribution over bids and corresponding distribution over prizes, and represent these distributions for all players simultaneously as a distribution over players, prizes, and bids. The result establishes the convergence of this distribution in weak*-topology as n grows large to a distribution over agent types, prizes, and bids that corresponds to a mechanism. This type of convergence enables us to approximate the average strategy (or the distribution over prize-bid pairs) of players that are close to a given agent type, but this may not be a good approximation of the equilibrium strategy of any single agent.

Under a strict single crossing condition on the agent's utility we establish a much stronger form of convergence, which delivers a simultaneous approximation of all players' equilibrium strategies. We show that in this case as n grows large players' equilibrium strategies become almost deterministic and the resulting allocation approaches the assortative one. In particular, this result applies to agent utilities that satisfy strict single crossing and are quasi-linear with respect to bids. For such utilities, in addition to the strong form of convergence to the assortative allocation, the IC-IR mechanism that implements the allocation is unique and can be characterized by applying standard techniques from the mechanism-design literature. We are therefore able to say (approximately, but with an arbitrary degree of precision as n increases) how each player will bid, and what prize she will obtain by making any given bid. This stronger result applies to many existing contest models, which are surveyed in Section 1.1.

Our results are useful in the analysis of large contests in three somewhat related ways. First, the results approximate the equilibria of contests whose exact equilibrium characterization is complicated. Second, they approximate the equilibria of contests for which there is no existing equilibrium characterization. These include a wide range of contests with heterogeneous prizes, contests that combine identical and heterogeneous prizes, and contests in which players' utilities are not quasi-linear in their bids. Third, the results facilitate contest design, welfare analysis, and comparative statics. They imply, for example, that under strict single crossing large contests approximately attain the unique assortative allocation.[1]

The rest of the paper is organized as follows. Section 1.1 surveys the related literature. Section 2 introduces the basic terminology and notation. Section 3 presents some examples

[1] A similar result was derived by Bulow and Levin (2006) in a non-trivial manner for a specific form of players' utilities (see Section 3.2).

that illustrate our results and the arguments that support them in concrete scenarios. The examples also describe important applications, which have been studied in the literature. Section 4 contains our main results and some discussion of their application. Section 5 contains the proofs of our main results when the limit set of prizes has full support. Section 6 concludes. Appendix A contains the proofs of intermediate results from Section 5. Appendix B contains the proofs of our main results when the limit set of prizes may not have full support. Appendix C shows that a similar approximation result holds for a contest in which players have head starts, even though our main results do not apply directly to this contest.

1.1 Related Literature

Our model includes many variants of the multi-prize all-pay auction with complete information, in which each player chooses a bid and pays the associated (and possibly idiosyncratic) cost.[2] Because the equilibria of such contests often have a complicated structure and are not straightforward to derive, existing closed-form equilibrium characterizations are limited to contests with either precisely two participating players and one prize (Hillman and Samet (1987), Hillman and Riley (1989), Che and Gale (1998, 2006), Kaplan and Wettstein (2006), Siegel (2010)), or identical prizes and players with identical costs (Baye, Kovenock, and de Vries (1993, 1996), González-Díaz (2012), Clark and Riis (1998)), or identical players (Barut and Kovenock (1998)). Equilibrium characterizations in the form of an algorithm are available for some contests with either identical prizes and players with heterogeneous costs (Siegel 2010, 2013a),[3] or with heterogeneous prizes and players with identical costs (Bulow and Levin (2006), González-Díaz and Siegel (2012), Xiao (2012)). The heterogeneity in prizes, however, is limited to very specific functional forms. Moreover, algorithmic

[2]Models of competition with incomplete information were studied by Amann and Leininger (1996), Moldovanu and Sela (2001, 2006), Parreiras and Rubinchik (2010), and Siegel (2013b) among others. Other models of competition postulate a probabilistic relation between competitors' efforts and prize allocation. See Tullock (1980) and Lazear and Rosen (1981). For a comprehensive treatment of the literature on competitions with sunk investments, see Nitzan (1994) and Konrad (2007).

[3]Siegel (2009) gives a closed-form expression for players' equilibrium payoffs, but does not solve for equilibrium.

characterizations make further analysis, such as comparative statics, difficult or impossible.

Our paper also contributes to the literature on large games. This literature typically makes continuity assumptions that exclude auction-like games (see, for example, Kalai (2004) and Carmona and Podczeck (2009, 2010)). A notable exception is Bodoh-Creed (2012), who explicitly considers uniform-price auctions with incomplete information, but assumes enough uncertainty about the set of prizes to exclude the possibility of a small change in the rank order of a bid having a large effect on the prize obtained. Moreover, the analysis in this literature often focuses on ε-equilibria of large games, which may not approximate Nash equilibria well. In contrast, our approach deals with the discontinuities that arise naturally in contests, approximates Nash equilibria, and uncovers a novel connection to mechanism design.

A more closely related paper is Hickman's (2009) theoretical analysis of affirmative action in college admissions. He considers a quasi-linear contest model with incomplete information that satisfies strict single crossing, and approximates the outcome for a large number of applicants by a continuum model in which the limit set of prizes has full support (so a small change in the rank order of a bid cannot have a large effect on the prize obtained). Our paper differs from Hickman's (2009) work in three main ways. First, our approach does not require quasi-linearity, strict single crossing, or full support of the limit prize set (although we obtain stronger results under these conditions), and is therefore applicable to a wide range of settings. Second, we relate the outcomes of large contests to single-agent mechanisms, which allows us, under certain conditions, to derive the approximation in closed form. Third, our model is one of complete information, and therefore includes many existing contest models as special cases.

From an applied perspective the assumption of complete information, which we interpret as competitors knowing the empirical distribution of their opponents, allows for a broad degree of ex-ante asymmetries among competitors, and fits settings in which data about the pool of competitors is available to the competitors (for example, colleges may release information about the composition of the applicant pool and firms may disclose aggregate information about the profile of their employees). The assumption of incomplete information, while suitable to many settings, implies that the ex-ante asymmetry among competitors is limited, and also implies that competitors entertain a small probability of facing an unreasonable set of competitors. From a theoretical perspective, complete information requires techniques that deal with non-monotonic, complicated mixed-strategy equilibria, which may also contain atoms and gaps, whereas incomplete information with strict single crossing leads to equilibria in strictly increasing pure strategies. Thus, techniques developed for incomplete-information settings are unlikely to apply to complete-information settings, whereas our approach seems readily modifiable to contests with incomplete information.

Finally, there is a literature on matching and search that employs continuum models to approximate matching models with many participants. Azevedo and Leshno (2012) show that stable matchings are easy to find and are often unique in a two-sided matching model with a continuum of agents on one side. Che and Kojima (2010) show that the random priority (random serial dictatorship) mechanism and the probabilistic serial mechanism converge to each other as the number of copies of a given set of object types grows large. Peters (2010) analyzes a model of directed search with a continuum of workers and firms, which is more tractable than the discrete version of the model.

2 Terminology and notation

2.1 Single-agent setting

There is a single agent of type $x \in X = [0, 1]$ with CDF F, where $F(x)$ is the mass of types z such that $z \leq x$.[4] It will sometimes be convenient to interpret the agent's types as a continuum of infinitesimal agents. There is a mass 1 of prizes $y \in Y = [0, 1]$ with CDF G, where $G(y)$ is the mass of prizes z such that $z \leq y$. We will sometimes refer to prize 0 as "no prize." We assume that F is continuous and strictly increasing, but G may be any probability distribution. As will become clear later, this choice of assumptions is motivated by applications. If G is strictly increasing on Y we say that G *has full support*, or that *prizes have full support*. Note that G may have full support even if there are masses of identical prizes, i.e., G is discontinuous, or, equivalently, G has atoms.

[4] All probability measures are defined on the σ-algebra of Borel sets.

The utility of the agent is given by a continuous function $U(x, y, t)$, where x is his type, y is the prize he obtains, and $t \geq 0$ is his bid. Since $y = 0$ corresponds to no prize, $U(x, 0, 0) = 0$ for all x. Higher prizes are better and higher bids are more costly, so $U(x, y, t)$ strictly increases in y for every $x > 0$ and $t \geq 0$, and strictly decreases in t for every $x \geq 0$ and $y \geq 0$. The utility is *quasi-linear* in bid if

$$U(x, y, t) = v(x, y) - t,$$

where $v(x, y)$ is the continuous utility of type x from obtaining prize y, and t is the monetary cost of bidding t.

We assume that sufficiently high bids are prohibitively costly, so $U(x, 1, b_{\max}) < 0$ for some $b_{\max}$ and all x. This occurs, for example, when the utility is quasi-linear. Thus, bids of $b_{\max}$ or higher are strictly dominated by 0, so we restrict the range of bids that the agent can make to $B = [0, b_{\max}]$. To simplify our proofs, we choose the number $b_{\max}$ to be rational.

We say that *weak single crossing* holds if for any $x_1 < x_2$, $t_1 < t_2$, and $y_1 < y_2$ we have that $U(x_1, y_2, t_2) \geq U(x_1, y_1, t_1)$ implies $U(x_2, y_2, t_2) \geq U(x_2, y_1, t_1)$. That is, if a lower type x_1 prefers to obtain a higher prize y_2 at a higher bid t_2 to obtaining a lower prize y_1 at a lower bid t_1, then so does any higher type x_2. If the higher type strictly prefers to obtain the higher prize at the higher bid, i.e., the second inequality is strict, then we say that *strict single crossing* holds.

A *consistent allocation* is a probability distribution H on $X \times Y$ whose marginal on X coincides with F and whose marginal on Y coincides with G. This condition says that precisely the available prizes are allocated to the agent types, and each type obtains exactly one prize (which can be equal to 0, that is, no prize). The conditional distribution H_x, $x \in X$, will be interpreted as the lottery over prizes that type x faces.

With quasi-linear utility, an allocation H is *efficient* if it allocates the prizes in a way that maximizes the non linear part of the agent's utility, i.e., it maximizes

$$\int_{x \in X} \int_{y \in Y} v(x, y) dH(x, y)$$

across all consistent allocations.

A (direct) *mechanism* M prescribes for each reported type $x \in X$ a joint probability distribution $Q_x(y, t)$ over prizes $y \in Y$ and bids $t \in B$. A mechanism is *incentive compatible*

(IC) if the expected utility of each type is maximized by reporting truthfully, i.e.,

$$\int_{y\in Y}\int_{t\in B} U(x,y,t)dQ_z(y,t)$$

is maximized at $z = x$.

A mechanism is *individually rational* (IR) if the expected utility of each type from reporting truthfully is at least as high as the utility from bidding 0 and obtaining the "lowest" available prize, i.e.,

$$\int_{y\in Y}\int_{t\in B} U(x,y,t)dQ_x(y,t) \geq U(x,z,0)\,, \tag{1}$$

where $z = \inf\{y : G(y) > 0\}$; in addition, we require that the inequality is an equality for at least one type x. If prizes have full support, then $z = 0$, so the right-hand side of (1) is $U(x,0,0) = 0$.

An *inverse tariff* is a non-decreasing upper semi-continuous function that maps bids to prizes; a *tariff mechanism* is an IR mechanism for which there exists an inverse tariff such that for every type $x \in X$ the probability distribution $Q_x(y,t)$ assigns probability 1 to the set of prize-bid pairs that maximize $U(x,y,t)$ among the prize-bid pairs in which the prize is the one assigned to the bid by the inverse tariff. A tariff mechanism is clearly IC.

A mechanism *implements* an allocation H if the marginal of Q_x on Y coincides with H_x for almost every x. This does not imply that H and $\{Q_x : x \in X\}$ determine a probability distribution on $X \times Y \times B$.[5] When they do, which will be the case in all our results, we say that the mechanism that implements H is *regular*, and refer to the distribution on $X \times Y \times B$ as the *outcome* of the mechanism.

2.2 Contests

Given a single-agent setting, for every n we define a complete-information contest ("the n-th contest") in which n players compete for n prizes (some prizes may be "no prize"). The players compete by each choosing a bid in B, simultaneously and independently. The player with the highest bid obtains prize n, the player with the second-highest bid obtains prize

[5] To see why, take some non-measurable function $f : X \to [0,\infty)$, have H distributed uniformly on, and assign probability 1 to, the diagonal $\{(x,x) : x \in X\}$, and have Q_x assign probability 1 to the pair $(x, f(x))$. That is, type x is prescribed bid $f(x)$ and prize x.

$n-1$, and so on. Ties are resolved by a fair lottery. The players and prizes correspond to the n-quantiles of the distributions of agent types and prizes in the single-agent setting. That is, the utility of player i from obtaining prize j and bidding t in the n-th contest is $U(x_i^n, y_j^n, t)$, where $x_i^n = F^{-1}(i/n)$ is the *player's type* (recall that F is continuous and strictly increasing) and $y_j^n = G^{-1}(j/n) = \inf\{z : G(z) \geq j/n\}$.[6] Note that if $j/n \leq G(0)$, then prize j is "no prize." A slight adaptation of the proof of Corollary 1 in Siegel (2009) shows that every contest has at least one Nash equilibrium.

To formalize our notions of approximation, for every n we transform any equilibrium of the n-th contest to a probability distribution D^n on $X \times Y \times B$, which we refer to as the *outcome* of the equilibrium. We then relate these distributions to probability distributions D that describe the outcomes of regular mechanisms. To begin, an equilibrium of the n-th contest determines for every player i a distribution over her bids, σ_i^n, and a distribution over the prizes she obtains for every bid, where we denote by $Q_i^n(j,t)$ her probability of obtaining prize j by bidding t.

We first define the conditionals of D^n on the sets $\{x_i^n\} \times Y \times \{t\}$ by setting the probability of obtaining prize y_j^n to $Q_i^n(j,t)$. We set the probability of obtaining any other prize y to 0. These conditionals and the distribution σ_i^n determine a distribution on $\{x_i^n\} \times Y \times B$. We define a distribution D^n on $X \times Y \times B$ by letting these distributions be the conditionals on the n sets $\{x_i^n\} \times Y \times B$, each of which has probability $1/n$, and assigning probability 0 to the complement of these sets.

Our results will show convergence of D^n to D. The notion of convergence, however, will vary across the results. We will introduce the appropriate notions when we formulate our results.

[6] The grid $1/n, \ldots, n/n$ can be replaced with any other n-element grid on $[0,1]$ such that the probability distribution that assigns probability $1/n$ to every element in the grid converges in **weak*-topology** to the uniform distribution on $[0,1]$.

3 Examples

We first demonstrate our approximation approach in a few contest settings that appeared in the literature.

3.1 All-pay auctions with identical prizes

This example focuses on all-pay auctions with identical prizes, which were studied by Clark and Riis (1998). They considered competitions for promotions, rent seeking, and rationing by waiting in line (see also Siegel (2010)).

For our single-agent setting assume a quasi-linear utility, a mass $P \in (0,1)$ of identical non-zero prizes, and agent types that coincide with the agent's utility for a non-zero prize. To satisfy continuity, set $v(x,y) = xy$ and have the support of G be $\{0,1\}$. (Note that this G does not have full support.) Assume a uniform type distribution, that is, $F(x) = x$ for all $x \in X$, and let $P = 1/2$, so $G(y) = 1/2$ for all $y \in [0,1)$ and $G(1) = 1$. We make these latter assumptions for ease of exposition.

An efficient allocation assigns prizes to the highest agent types; this is done, for example, by the probability measure H on $X \times Y$ that assigns probability 1 to, and is distributed uniformly on, the set

$$\{(x,0) : x \leq 1/2\} \cup \{(x,1) : x > 1/2\}.$$

The unique IC-IR mechanism that implements this allocation prescribes for every type $x \leq 1/2$ bid 0 and for every type $x > 1/2$ bid $1/2$. The mechanism is a tariff mechanism with a discontinuous (but upper semi-continuous) inverse tariff that maps bids $t \in [0, 1/2)$ to prize 0 and bids $t \geq 1/2$ to prize 1. Indeed, given the inverse tariff, the optimal bid is 0 for types $x < 1/2$ and $1/2$ for types $x > 1/2$. For type $1/2$ both 0 and $1/2$ are optimal. Therefore, the mechanism prescribes an optimal bid for every type x. In this example, and in the examples discussed later, it is easy to see that the IC-IR mechanisms that implement efficient allocations are regular.

We now show that the outcome of the mechanism approximates the equilibria of contests with many players and prizes. Clark and Riis (1998) solved for the unique equilibrium in this setting and describe it in closed form, and we use their characterization to demonstrate the approximation.[7] For every n, the n-th contest is an all-pay auction with n players and $m \equiv \ulcorner n/2 \urcorner$ (non-zero) prizes, and the value of a non-zero prize to player i is i/n.

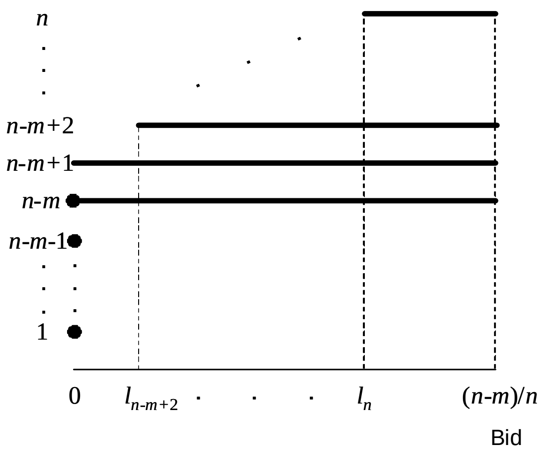

Figure 1: The support of players' strategies (dots represent atoms) in the unique equilibrium

As Figure 1 shows, in equilibrium the $n-m-1$ players with the lowest valuations bid 0, and each of the $m+1$ players with the highest valuations bids on an interval. The lower bounds of these intervals are monotonic in players' valuations, and the common upper bound is $(n-m)/n = 1 - m/n \in \{1/2, 1/2 + 1/2n\}$. By (2) in Clark and Riis (1998), the lower bound of the bidding interval of player $i > n-m$ is

$$l_i = \left(1 - \frac{m}{n}\right)\left(1 - \Pi_{k=n-m+1}^{i} \frac{k}{i}\right). \tag{2}$$

This implies that for any $N > 0$, as n grows large the lower bound of the bidding interval of player $i = n-m+N$ (and therefore also those of players $m-n, \ldots, m-n+N-1$) approaches 0. Thus, for every $\varepsilon > 0$, as n grows large the number of players with valuations greater than 1/2 who bid on $[\varepsilon, 1-m/n]$ grows large. This contrasts with the previous observation

[7] An alternative approach, in the spirit of our general results, is only to partially characterize the equilibria for large n, using equilibrium properties derived from first principles, and use this partial characterization to establish the approximation. We omit this derivation for the example in the interest of brevity; it is available from the authors upon request.

that in the single-agent mechanism types higher than 1/2 bid 1/2, so it may seem that the outcome of the mechanism does not approximate the equilibrium for large n. This apparent discrepancy is overcome by noting that the lower bound of the bidding interval of a player with valuation approximately $(1/2)+\varepsilon$ is for large n approximately

$$\frac{1}{2}\left(1-\frac{\frac{1}{2}\cdot\left(\frac{1}{2}+\frac{1}{n}\right)\cdot\ldots\cdot\left(\frac{1}{2}+\frac{\varepsilon n}{n}\right)}{\left(\frac{1}{2}+\varepsilon\right)^{\varepsilon n}}\right)=\frac{1}{2}\left(1-\frac{\frac{1}{2}\cdot\ldots\cdot\left(\frac{1}{2}+\frac{\varepsilon n}{2n}\right)}{\left(\frac{1}{2}+\varepsilon\right)^{\frac{\varepsilon n}{2}}}\cdot\frac{\left(\frac{1}{2}+\frac{\varepsilon n}{2n}+\frac{1}{n}\right)\cdot\ldots\cdot\left(\frac{1}{2}+\frac{\varepsilon n}{n}\right)}{\left(\frac{1}{2}+\varepsilon\right)^{\frac{\varepsilon n}{2}}}\right).$$

The first fraction is bounded above by

$$\left(\frac{\frac{1}{2}+\frac{\varepsilon}{2}}{\frac{1}{2}+\varepsilon}\right)^{\frac{\varepsilon n}{2}}\underset{n\to\infty}{\rightarrow}0,$$

and the second fraction is bounded above by 1, so as n increases the lower bound of the bidding interval approaches 1/2. Therefore, for any $\varepsilon>0$, for sufficiently large n at most a fraction ε of the players bid more than ε away from what the mechanism prescribes for the types that correspond to them.

3.2 All-pay auctions with heterogeneous prizes and multiplicative utilities

This example focuses on all-pay auctions with heterogeneous prizes and multiplicative utilities, which were studied by Bulow and Levin (2006), hence forth B&L. They considered the national residency matching program, in which residents are matched to hospitals. Hospitals compete by offering identity-independent wages, each hospital can hire one resident, and hospital i's utility from hiring resident j is ij.

For our single-agent setting assume a quasi-linear utility, a continuum of different prizes, and a utility for a prize that is the product of the agent's type and the prize. Thus, $v(x,y)=xy$. We assume that F and G are uniform.[8] (Note that prizes have full support.)

The assortative allocation, in which type x obtains prize x, is efficient. An IC mechanism that implements this allocation and prescribes for type x bid $t(x)$ has the property that

$$x=\arg\max_{z}\{xz-t(z)\}.$$

[8] B&L make this assumption in Section 7, in which they study the contest for $n\to\infty$.

This implies that $t'(x) = x$. Since in an IR mechanism the bid of type 0 has to be 0 (because type 0 obtains prize 0), we have that $t(x) = x^2/2$. This mechanism is a tariff mechanism with a continuous inverse tariff that maps every bid $t \in [0, 1]$ to prize $\sqrt{2t}$. Indeed, given the inverse tariff, the optimal bid for type x is $x^2/2$, which is the bid the mechanism prescribes for type x.

We now provide some intuition for why the outcome of this mechanism approximates the equilibria of contests with many players and prizes. For every n, the n-th contest is an all-pay auction with n players and n prizes, and the value of prize j to player i is ij/n^2.[9] As in the previous example, the simplicity of the allocation and the IC-IR mechanism that implements it contrasts with the relative complexity of players' equilibrium mixed strategies, which are depicted in Figure 2. Moreover, while the IC-IR mechanism is given explicitly, players' mixed strategies are derived by an algorithm and are not described in closed form. We therefore only provide a heuristic argument for the approximation. The argument makes use of some equilibrium properties demonstrated by B&L, but does not require their full algorithm for constructing the equilibrium. A complete, but somewhat less illuminating, proof of the approximation can be obtained by adapting the proof of Theorem 1 below.

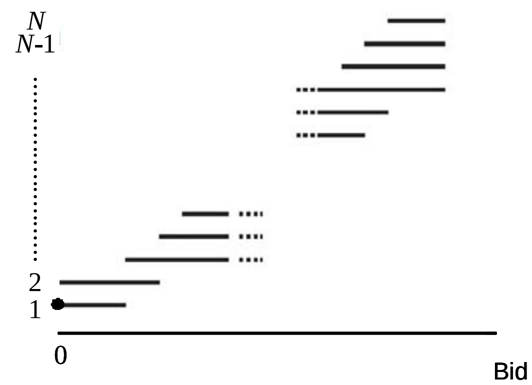

Figure 2: The support of players' strategies in the unique equilibrium

The outline of the argument is as follows. Each player chooses a bid from an interval, and the intervals are staggered so that the intervals of higher players have higher lower and

[9]In the notation of B&L, $\Delta_i = i/n^2$, as in their Proposition 5.

upper bounds. The number of players that have a given bid in their interval is known. This enable us to divide each bidding interval to a known number of smaller intervals, which turn out to be of approximately equal length, such that the density on each subinterval is known and constant. This enables us to compute the length of the interval and tells us how this length changes across players. This also shows that the intervals shrink to points as n grows large. Taken together, these observations imply efficiency and pin down the limiting bid of each player.[10]

We now describe the argument in greater detail. B&L show that player i' s strategy is continuously distributed on an interval $[b_i^n, d_i^n]$. The intervals have the property that $b_i^n < b_j^n$ and $d_i^n \leq d_j^n$ for any $i < j$ (except $i = 1$ and $j = 2$, in which case $b_i^n = b_j^n = 0$).

In particular, if a bid t is contained in some player's bidding interval, then it is contained in the bidding intervals of players $l(m)$, $l(m)+1$,..., m, where $l(m)$ is the lowest player whose interval contains t, and m is the highest player whose interval contains t. B&L show that

$$l(m) = \arg\min_l \left\{ \frac{1}{m-l} \sum_{k=l}^{m} \frac{n^2}{k} - \frac{n^2}{l} > 0 \right\}, \tag{3}$$

and that the density of the strategy of player l, $l(m) \leq l \leq m$, at bid t that belongs to her bidding interval is

$$\frac{1}{m-l(m)} \sum_{k=l(m)}^{m} \frac{n^2}{k} - \frac{n^2}{l}. \tag{4}$$

(see their Lemma 2 and the following paragraph).

Choose a rational $x \in (0,1)$, and take a sequence of $n, m \to \infty$ such that $x = m/n$. Partition the bidding interval of player m into subintervals such that player m is the lowest bidder on the rightmost subinterval, the second lowest bidder on the second rightmost subinterval, and so on, until the leftmost subinterval, on which she is the highest bidder. It follows from (3) that $m - l(m)$ differs from $\sqrt{2l(m)}$ by at most 1 (see their Lemma 3). In particular, $l(m)$ is of order m. Therefore, the number of subintervals in the partition is approximately $\sqrt{2m}$.

From (4), the density of player m's strategy on these subintervals is approximately

$$d, d + \frac{n^2}{l(m)[l(m)+1]}, d + \frac{2n^2}{l(m)[l(m)+2]}, ..., d + \frac{\sqrt{2m}n^2}{l(m)[l(m)+\sqrt{2m}]},$$

[10] B&L demonstrated that virtually all the surplus is realized as n grows large.

respectively, where d is the density on the rightmost subinterval. Moreover, (3) implies that d cannot exceed $n^2/l(m)[l(m)-1]$.[11]

The lengths of these subintervals are approximately equal.[12] Denote this common length by Δ. Since

$$d\Delta + \left[d + \frac{n^2}{l(m)[l(m)+1]}\right]\Delta + \left[d + \frac{2n^2}{l(m)[l(m)+2]}\right]\Delta + ... + \left[d + \frac{\sqrt{2m}n^2}{l(m)[l(m)+\sqrt{2m}]}\right]\Delta \approx 1$$

and $l(m)$ is of order m, we have that Δ is of order x^2/m.

This implies that the length of each bidding interval tends to 0 as n and m grow large. Passing to a subsequence if necessary, these shrinking intervals converge to a number $t(x)$.[13] Since $t(x)$ and $t(x-1/n)$ differ approximately by the length of one interval, Δ, the derivative of $t(x)$ at x is

$$t'(x) = \frac{\Delta}{1/n} = x. \tag{5}$$

In particular, higher types obtain higher prizes in the limit. This implies that type x obtains prize x. Since type 0 obtains 0, (5) implies that type x bids $t(x) = x^2/2$.

---

[11] Indeed,

$$\begin{aligned}\frac{m-l(m)}{m-l(m)+1}d &= \frac{1}{m-l(m)+1}\sum_{k=l-1}^{m}\frac{n^2}{k} - \frac{n^2}{l-1} \\ &- \frac{1}{m-l(m)+1}\frac{n^2}{l-1} - \frac{m-l(m)}{m-l(m)+1}\frac{n^2}{l-1} + \frac{n^2}{l} \\ &= \frac{1}{m-l(m)+1}\sum_{k=l-1}^{m}\frac{n^2}{k} - \frac{n^2}{l-1} \\ &+ \frac{m-l(m)}{m-l(m)+1}\frac{n^2}{[l(m)-1]l(m)} \leq \frac{m-l(m)}{m-l(m)+1}\frac{n^2}{[l(m)-1]l(m)}.\end{aligned}$$

The last inequality follows from the definition of $l(m)$. This yields

$$d \leq \frac{n^2}{[l(m)-1]l(m)},$$

as required.

[12] Intuitively, this follows from Lemma 3 in B&L; or, more precisely, from the fact that $m - l(m)$ is of order $\sqrt{2l(m)}$. We omit the details of a formal proof.

[13] In this heuristic argument, we do not show that $t(x)$ is independent of the choice of this subsequence. (This independece also implies that there is no need to pass to a subsequence.)

3.3 All-pay auctions with heterogeneous prizes and identical players

This example focuses on all-pay auctions with heterogeneous prizes and identical players, which were studied by Barut and Kovenock (1998). They considered grading, promotions, procurement settings, and political competitions.

For our single-agent setting, assume a quasi-linear utility, a continuum of different prizes, and a utility for a prize that is independent of the agent's type. Thus, $v(x, y) = y$. For ease of exposition, we assume that F and G are uniform.

All allocations are efficient. One example is the uniform allocation, whose density is $h(x, y) = 1$ for all values of x and y. The unique IC-IR mechanism that implements this allocation has $Q_x(y, t)$ distributed uniformly on the diagonal $y = t$. This is a tariff mechanism with a continuous inverse tariff that maps every bid $t \in [0, 1]$ to prize t. Indeed, given the inverse tariff, all bids in $[0, 1]$ are optimal for every type.

We now show that the outcome of this mechanism approximates the equilibria of contests with many players and prizes. For every n, the n-th contest is an all-pay auction with n players and n prizes, and the value of prize j to every player is j/n. Barut and Kovenock (1998) showed that the contest has a unique equilibrium, in which all players randomize uniformly across all bids $t \in [0, 1]$. Thus, the players behave in the contest exactly as the types that correspond to them do in the mechanism. Moreover, by the law of large numbers, for every $\varepsilon > 0$ and large enough n, a player who bids y obtains a prize in $[y - \varepsilon, y + \varepsilon]$ with probability at least $1 - \varepsilon$.

4 Main results

4.1 Results with strict single crossing

We begin with our stronger results, which apply to settings in which strict single crossing holds. In such settings, any tariff mechanism has the property that higher types are prescribed weakly higher prizes. This motivates our definition of the *assortative allocation*, in which type $x > 0$ obtains prize $y = G^{-1}(F(x))$, where $G^{-1}(z) = \inf\{y : G(y) \geq z\}$, and type 0 obtains prize $\inf\{y : G(y) > 0\}$. The assortative allocation is obviously consistent. Our results show that under strict single crossing the equilibria of large contests are approximated by tariff mechanisms that implement the assortative allocation. We first formulate our results for settings in which prizes have full support, and then for settings in which this condition is violated. The condition that G has full support guarantees that G^{-1} is continuous, so when the number of players and prizes is large, it is enough to know the approximate rank-order of a player's bid to know the approximate utility she derives from the prize she obtains. For the formulation of the results, recall that $x_i^n = F^{-1}(i/n)$.

Theorem 1 *Suppose that strict single crossing holds and prizes have full support. Then, for any $\varepsilon > 0$ and any equilibrium of a sufficiently large contest,*

(a) with probability 1 every player i obtains a prize that differs by at most ε from $G^{-1}(F(x_i^n))$;

(b) there is a regular tariff mechanism with a continuous inverse tariff that implements the assortative allocation, such that with probability 1 the bid of every player i differs by at most ε from the bid that the mechanism prescribes for type x_i^n.

When there is a unique IC-IR mechanism that implements the assortative allocation,[14] the mechanism in part (b) of Theorem 1 coincides with this mechanism.

If U is quasi-linear, then the assortative allocation is also efficient, because $v_{x_2}(y_2) - v_{x_1}(y_2) > v_{x_2}(y_1) - v_{x_1}(y_1)$ for any $x_1 < x_2$ and $y_1 < y_2$.[15] If, in addition, v satisfies the conditions of the envelope theorem, then the unique IC-IR mechanism that implements the

[14] This happens, for example, if U, F, and G are smooth functions, the partial derivative of U with respect to t is bounded away from 0, and the solution to the problem

$$\max_z U(x, G^{-1}F(z), t(z))$$

is determined by the first-order condition. In this case

$$t'(x) = \frac{\partial U/\partial y(x, G^{-1}F(x), t(x))}{\partial U/\partial t(x, G^{-1}F(x), t(x))}(G^{-1}F)'(x)$$

by the envelope theorem, and this differential equation has a unique solution that satisfies the boundary condition $t(0) = 0$.

[15] To see why, apply strict single crossing to $t_1 = v_{x_1}(y_1)$ and $t_2 = v_{x_1}(y_2)$.

assortative allocation prescribes for type x bid

$$br(x) = v\left(x, G^{-1}(F(x))\right) - \int_0^x v_x\left(z, G^{-1}(F(z))\right) dz. \tag{6}$$

We therefore have the following corollary of Theorem 1.

Corollary 1 *Suppose that strict single crossing holds, prizes have full support, U is quasi-linear, and v satisfies the conditions of the envelope theorem. Then, for any $\varepsilon > 0$ and any equilibrium of a sufficiently large contest, with probability 1 every player i bids within ε of $br(x_i^n)$ given by (6) and obtains a prize that differs by at most ε from $G^{-1}(F(x_i^n))$.*

Corollary 1 applies, for example, to the setting of Section 3.2.

We now consider settings in which strict single crossing holds, but prizes may not have full support. This is the case in the setting of Section 3.1. The difficulty when G does not have full support is that G^{-1} is discontinuous, so the approximate rank-order of a player's bid may be insufficient to know the approximate utility she derives from the prize she obtains. In the setting of Section 3.1, for example, if the rank-order of a player's bid is approximately $n/2$, we do not know whether she obtains a non-zero prize or not. Nevertheless, the following result shows that the approximation of Theorem 1 holds "up to ε."

Theorem 2 *Suppose that strict single crossing holds and prizes have full support. Then, for any $\varepsilon > 0$ and any equilibrium of a sufficiently large contest,*

(a) a fraction of at least $1-\varepsilon$ of the players i obtain with probability at least $1-\varepsilon$ a prize that differs by at most ε from $G^{-1}(F(x_i^n))$;

(b) there is a regular tariff mechanism that implements the assortative allocation, such that the bid of each of a fraction of at least $1-\varepsilon$ of the players i differs with probability at least $1-\varepsilon$ by at most ε from the bid that the mechanism prescribes for type x_i^n.

We also have an analogue of Corollary 1.

Corollary 2 *Suppose that strict single crossing holds, U is quasi-linear, and v satisfies the conditions of the envelope theorem. Then, for any $\varepsilon > 0$ and any equilibrium of a sufficiently large contest, each of a fraction of at least $1-\varepsilon$ of the players i bids with probability at least $1-\varepsilon$ within ε of $br(x_i^n)$ given by (6) and obtains with probability at least $1-\varepsilon$ a prize that differs by at most ε from $G^{-1}(F(x_i^n))$.*

Corollary 2 applies, for example, to the setting of Section 3.1.

In addition to approximating the outcome of large contests that have been solved in the literature, Theorems 1 and 2 and their corollaries apply to many contests for which there is no existing equilibrium characterization. To demonstrate this, consider the following example.

Suppose that F and G are uniform, so the assortative allocation assigns prize x to type x. Let the agent' utility be quasi-linear with $v(x,y) = xh(y)$ for some strictly increasing and continuous function h such that $h(0) = 0$. As we saw in Section 3.2, the case $h(y) = y$ corresponds to the setting of B&L. The case $h(y) = y^2$ corresponds to Xiao's (2013) quadratic prize sequence.[16] The case $h(y) = e^y$ corresponds to Xiao's (2013) geometric prize sequence.[17] Xiao (2013) shows that players' equilibrium strategies in these two cases may contain gaps (see Figure 3). This does not happen in the linear setting of B&L. For contests that correspond to other, non-trivial functions h (including $h(y) = y^m$ for $m > 2$), there is no existing equilibrium characterization.

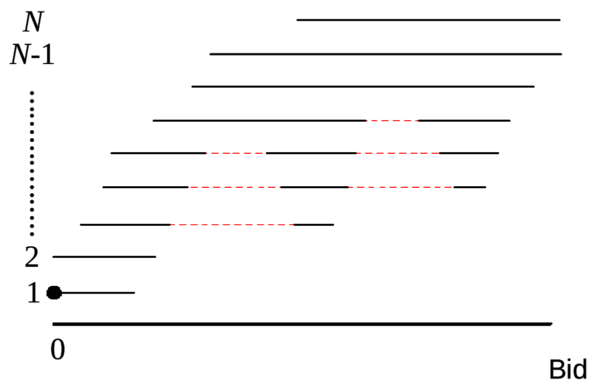

Figure 3: The support of players' equilibrium strategies (dotted lines represent gaps) in a possible equilibrium in the setting of Xiao (2013)

Since for any strictly increasing h strict single crossing holds, Corollary 1 applies, and (6) shows that $br(x) = xh(x) - \int_0^x h(y)\,dy$. For large n, therefore, a player with type x bids

[16] This is because $v(x,(j+1)/n) - v(x,j/n) = x(2j+1)/n^2$, so

$$v\left(x,\frac{j+1}{n}\right) - v\left(x,\frac{j}{n}\right) - \left(v\left(x,\frac{j}{n}\right) - v\left(x,\frac{j-1}{n}\right)\right) = x\frac{2j+1}{n^2} - x\frac{2j-1}{n^2} = x\frac{2}{n^2}.$$

[17] This is because $v(x,(j+1)/n)/v(x,j/n) = xe^{1/n}$.

close to $x - \int_0^x h(y)\,dy$ and obtains a prize close to x.

Theorems 1 and 2 and their corollaries also apply to contests that combine identical (non-zero) and heterogeneous prizes, which have not been studied in the literature. Such contests correspond to distributions G that have atoms. For example, let

$$G(y) = \begin{cases} y/2 & 0 \leq y < 1 \\ 1 & y \geq 1 \end{cases},$$

so that half the prizes have value 1. Letting $h(y) = y$, we obtain a setting similar to that of Section 3.2, which differs from that example in that there are many identical best prizes. In this case, the assortative allocation assigns prize $2x$ to every type $x \leq 1/2$, and prize 1 to every type $x > 1/2$. Corollary 1 applies, and (6) shows that $br(x) = x^2$ for $x < 1/2$ and $br(x) = 1/4$ for $x \geq 1/4$. For large n, therefore, a player with type x bids close to $\min\{x^2, 1/4\}$ and obtains a prize close to $\min\{2x, 1\}$.

In addition, Theorems 1 and 2 and their corollaries apply to settings in which $v(x, y)$ is not multiplicatively separable, and Theorems 1 and 2 apply to settings in which $U(x, y, t)$ is not quasi-linear in t.

4.2 Results without strict single crossing

We now turn to settings in which strict single crossing may not hold. Without strict single crossing, an approximating mechanism may implement a consistent allocation other than the assortative one, and may approximate large contests not as well as under strict single crossing.

Theorem 3 *Regardless of single crossing (or prizes having full support), for any $\varepsilon > 0$, any metrization of weak*-topology,*[18] *and any equilibrium of a sufficiently large contest there is a regular tariff mechanism that implements a consistent allocation whose outcome is ε-close (in the metrization of weak*-topology) to the outcome of the equilibrium. If prizes have full support, then the inverse tariff is continuous. If weak single crossing holds and the utility is quasi-linear, then the consistent allocation is also efficient.*

[18] See Rudin (1973) for the definition of this topology and its properties.

Theorem 3 applies to the setting of Section 3.3 (including the implications when prizes having full support and weak single crossing holds), whereas Theorems 1 and 2 do not.

The approximation in Theorem 3 is weaker than those in the previous results. In fact, it may be that even for large n the approximating mechanism does not approximate the behavior of any player individually. To illustrate this, consider the setting of Section 3.1 with $v(x, y) = 1$ instead of $v(x, y) = x$, and let $n = 2k + 1$. Then the n-th contest has an equilibrium in which the k even players $2, 4, \ldots, 2k$ bid 0 and obtain no prize, and the $k + 1$ odd players $1, 3, \ldots, 2k + 1$ employ the same mixed strategy on $[0, 1]$. As n increases, the mixing players bid close to 1 and obtain a prize with probability close to 1. Therefore, the distributions D^n converge in weak*-topology to the distribution D in which every type x bids 0 and obtains 0 with probability $1/2$ and bids 1 and obtains 1 with probability $1/2$. Consequently, the strategies of all players in every contest qualitatively differ from the conditionals of D.

Nevertheless, for every type x Theorem 3 allows us to approximate, for large n, the joint equilibrium distribution over prize-bid pairs of players with types close to x. To see why, take a closed interval I that contains x, a slightly larger open interval J, and a continuous function $f : X \times Y \times B \to [0, 1]$ whose value is 1 on $I \times Y \times B$ and 0 on the complement of $J \times Y \times B$. Then

$$R(y, t) = \int_{I\times[0,y)\times[0,t)} f(x, z, s)dD \Big/ \int_{I\times Y\times B} f(x, z, s)dD$$

can be interpreted as a distribution over prize-bid pairs that approximates, for large n, the joint equilibrium distribution over prize-bid pairs of players with types close to x. The reason that this approximation works while the approximation of each player's equilibrium distribution may fail is that in order to obtain an approximation for players with types in a neighborhood of a given type x, we need to take a sufficiently large n. Then, the neighborhood may include many other player types, and from the limit distribution we only learn about the average strategy across all these types.

If, however, for every type x there is a unique prize-bid pair (y_x, t_x) that maximizes $U(x, y, t)$ among the prize-bid pairs in which the prize is the one assigned to the bid by the inverse tariff, then convergence in weak*-topology implies convergence in a sense similar to that of Theorem 2. Namely, for any $\varepsilon > 0$ and sufficiently large n, for a fraction $1-\varepsilon$ of the players, with probability $1-\varepsilon$ the prize that player i obtains differs from $y_{x_i^n}$ by at most ε and the bid of player i differs from $t_{x_i^n}$ by at most ε.

Another difference between Theorem 3 and the previous results is that the set of tariff mechanisms that implement a consistent allocation may be quite large even if every contest has a unique equilibrium. For example, in the setting of Section 3.3 there is a continuum of efficient allocations and tariff mechanisms that implement them, all associated with the same inverse tariff that maps every bid $t \in [0,1]$ to prize t. We conjecture, however, that Theorem 3 is the strongest general convergence result that one can obtain. This is because some contests have many equilibria, and different sequences of equilibria may be approximated by different mechanisms. To see this, consider again the setting of Section 3.1 with $n = 2k+1$ and $v(x,y) = 1$ instead of $v(x,y) = x$. The n-th contest has an equilibrium in which players $1, \ldots, k$ bid 0 and obtain no prize, and players $k+1, \ldots, n$ employ the same mixed strategy on $[0,1]$. As n increases, the mixing players bid close to 1 and obtain a prize with probability close to 1. Therefore, the distributions D^n converge in weak*-topology to the distribution D in which every type $x < 1/2$ bids 0 and obtains no prize and every type $x \geq 1/2$ bids 1 and obtains a prize. Similarly, there is a sequence of equilibria with a limit distribution in which every type $x \leq 1/2$ bids 1 and obtains a prize and every type $x > 1/2$ bids 0 and obtains no prize, as well as many other sequences of equilibria with different limit distributions.

5 Proofs

As a preliminary step, we choose an equilibrium for each contest, and refer to the sequence in which the n-th element is the equilibrium of the n-th contest as the *sequence of equilibria*. For each of the theorems, we will show that every subsequence of this sequence contains a further subsequence that satisfies the statement of the theorem. This suffices, because the following observation can be applied with Z_n being the set of equilibria of contest n.

(**Subsequence Property**) Given a sequence of sets $\{Z_n : n = 1, 2, \ldots\}$, suppose that for every sequence $\{z_n : n = 1, 2, \ldots\}$ with $z_n \in Z_n$ it is the case every subsequence $\{z_{n_k} : k = 1, 2, \ldots\}$ contains a further subsequence $\{z_{n_{k_l}} : l = 1, 2, ...\}$ such that every element $z_{n_{k_l}}$ has some property. Then there exists an N such that for every $n \geq N$ every element in Z_n has this property.[19]

The proof of Theorem 2 is in Appendix B. The structure of the proof is similar to that of Theorem 1 below, but the proof of Theorem 1 relies heavily on the continuity of G^{-1}, so to prove Theorem 2 almost every step of the proof must be modified and additional results must be derived.

5.1 Proof of Theorem 1

We begin with an outline of the proof. Given a subsequence of equilibria, each equilibrium in the subsequence induces for each player a mapping from bids to expected percentile rankings. We consider the average of those mappings, and G^{-1} composed with this average gives a mapping T^n from bids to prizes. As n increases, this mapping approximates the equilibrium mappings from bids to prizes of all players in the n-th contest. We then find a subsequence of T^n that converges to some limit mapping T from bids to prizes. To do this, we first define T on the rationals in B by successively picking subsequences that are convergent on an increasing number of the rationals and applying a diagonalization procedure. Next, we extend the resulting function to B. We show that the resulting T is continuous and the subsequence of T^n converges uniformly to T. Then, for each agent type we define the set of optimal bids when T is treated as an inverse tariff, and define the mapping from agent types to sets of optimal bids. We consider a small neighborhood of the graph of this mapping, and show that for large n every player i's best responses in the n-th contest are in the "x_i^n-slice" of this object. This slice is a superset of a small neighborhood of the set of optimal bids of the agent type. In particular, a slice can in principle be large even if the set of optimal bids of the corresponding agent type is small (this happens when the set of optimal bids of a nearby agent type is large). We show, however, that strict single crossing combined with this characterization of players' best responses implies that each agent type has a single optimal bid, which is continuous and weakly increases in the agent type. This implies that the characterization bounds every player's best response set, and therefore the support

[19] Otherwise, there would be a subsequence of elements without the property.

of her equilibrium strategy, within an arbitrarily small interval as n increases. We then conclude that the unique mechanism induced by T implements the assortative allocation. This demonstrates part (b) in the statement of the theorem; part (a) then follows easily.

To simplify notation, we take the subsequence to be the sequence of equilibria (this has no effect on the proof). We denote an equilibrium by $\sigma^n = (\sigma_1^n, \ldots, \sigma_n^n)$, where σ_i^n is player i's equilibrium strategy in the n-th contest (we abuse notation by using σ_i^n sometimes to refer to the random variable whose range is B and sometimes to refer to the induced distribution on B). We denote by $R_i^n(t)$ the random variable that is the percentile location of player i in the ordinal ranking of the players in the n-th contest if she bids slightly above t and the other players employ their equilibrium strategies.[20] That is,

$$R_i^n(t) = \frac{1}{n}\left(1 + \sum_{k \neq i} 1_t(\sigma_k^n)\right),$$

where $1_t(\sigma)$ is 1 if $\sigma \leq t$ and 0 otherwise. Let

$$A_i^n(t) = \frac{1}{n}\left(1 + \sum_{k \neq i} \Pr(\sigma_k^n \leq t)\right)$$

be the expected percentile ranking of player i. Then, by Hoeffding's inequality, for all t in B we have

$$\Pr(|R_i^n(t) - A_i^n(t)| > \delta) < 2\exp\left\{-2\delta^2(n-1)\right\}. \tag{7}$$

Finally, let

$$A^n(t) = \frac{1}{n}\sum_{i=1}^{n} A_i^n(t)$$

be the average of the expected percentiles rankings of the players in the n-th contest if they bid t and the other players employ their equilibrium strategies.

Now, let T^n be the mapping from bids to prizes induced by A^n. That is, $T^n(t) = G^{-1}(A^n(t))$. Recall that $G^{-1}(z) = \inf\{y : G(y) \geq z\}$ and that G^{-1} is continuous (prizes have full support).

Because A^n is (weakly) increasing, so is T^n. Take an ordering of all rationals in B, denoted by $q_1, q_2, \ldots$. Take a converging subsequence of the sequence $T^n(q_1)$, denote it by

[20] This is the infimum of her ranking if she bids above t, which is equivalent to bidding t and winning any ties there. If players' strategies are continuous, then this is equivalent to bidding t.

$T^{n_1}(q_1)$, and denote its limit by $T(q_1)$. Take a converging subsequence of the sequence $T^{n_1}(q_2)$, denote it by $T^{n_2}(q_2)$, and denote its limit by $T(q_2)$. Continue in this fashion to obtain a function $T : \{q_1, q_2, \ldots\} \to [0, 1]$. In addition, define a subsequence of T^n such that its k-th element is the k-th element in the sequence T^{n_k}. For the rest of the proof, denote this new sequence by T^n.

We now describe some properties of T:

(1) T is (weakly) increasing, because every T^n is (weakly) increasing.

(2) $T(0) = 0$.

Indeed, suppose to the contrary that $T(0) > 0$. This implies that for some $\delta > 0$ and large enough n, we would have that $A^n(0) > G(0) + \delta$. This means, in turn, that the strategies of a fraction of at least $G(0) + \delta$ players in the n-th contest have atoms at 0. Therefore, there is a positive probability that these players tie for a fraction δ of prizes of positive value, so any one of them would be better off bidding slightly above 0 instead of bidding 0.

(3) $T(b_{\max}) = 1$, because $A^n(b_{\max}) = 1$ and therefore $T^n(b_{\max}) = 1$.

The following lemma shows that T can be extended uniquely to a continuous function on the entire interval B.

Lemma 1 *For any $t \in B$ (not necessarily rational) and any two sequences $q^m \uparrow t$ and $r^m \downarrow t$ of rationals in B, we have $\lim T(q^m) = \lim T(r^m)$.*

The proof of Lemma 1, and those of other results in this subsection, is in Appendix A. The idea of the proof is that if T is discontinuous at some t, then for large n it is better to bid slightly above t than slightly below t. This violates the following property, which we also use in other proofs.

(**No-Gap Property**) In any equilibrium, there is no interval $(a, b) \in B$ of positive length in which all players bid with probability 0 and some player bids in $[b, b_{\max}]$ with positive probability.

Indeed, suppose the contrary, and consider such a maximal interval (a, b). A player would only bid b or slightly higher than b if some other player has an atom at b. But the player with the atom would be better off by either slightly increasing her bid (if she is tying with other players and winning the tie would increase her prize) or by decreasing her bid (in the complementary case).

We extend T to the entire interval B by setting $T(t) = \lim T(t^m)$ for some sequence $t^m \to t$ of rationals in B. Lemma 1 shows that $T(t)$ is the same regardless of the chosen sequence t^m. Lemma 1 also guarantees that this is indeed an extension, and that the extended T is continuous. Continuity implies the following result.

Lemma 2 *T^n converges to T uniformly on B.*

We now relate players' equilibrium behavior in large contests to the inverse tariff T. For every x, denote by BR_x type x's set of optimal bids given T, i.e., the bids t that maximize $U(x, T(t), t)$. Denote by $BR(\varepsilon)$ the ε-neighborhood of the graph of the correspondence that assigns to every $x \in [0,1]$ the set BR_x, i.e., $BR(\varepsilon)$ is the union over all types x and bids $t \in BR_x$ of the open balls of radius ε centered at (x,t). For every type x denote by $BR_x(\varepsilon)$ the set of bids t such that $(x,t) \in BR(\varepsilon)$.

Note that $BR(\varepsilon)$ is a 2-dimensional open set, while each $BR_x(\varepsilon)$ is a 1-dimensional "slice." Note also that $BR(\varepsilon)$ is in general larger than the union across x of the pairs (x,t) for which the distance between t and some bid in BR_x is less than ε. In particular, $BR_x(\varepsilon)$ may contain bids whose distance from every bid in BR_x is more than ε. Using the sets $BR_x(\varepsilon)$, we can characterize players' equilibrium behavior.

Lemma 3 *For every $\varepsilon > 0$, there is an N such that for every $n \geq N$, the equilibrium bid of every player $i = 1, ..., n$ in the n-th contest belongs to $BR_{x_i^n}(\varepsilon)$ with probability 1, i.e.,*

$$\sigma_i^n(BR_{x_i^n}(\varepsilon)) = 1.$$

The proof up to this point did not rely on strict single crossing, a fact we will use in the proof of Theorem 3. We now show that under strict single crossing BR_x is a singleton.

Lemma 4 *If strict single crossing holds, then for all x the set BR_x is a singleton. In addition, the function br that assigns to x the single element of BR_x is continuous and weakly increasing.*

Lemma 4 implies that for every $\varepsilon > 0$ there is a $\delta > 0$ such that

$$BR(\delta) \subseteq \cup_{x \in X} \left(x \times [br(x) - \varepsilon, br(x) + \varepsilon]\right),$$

i.e., $BR_x(\delta) \subseteq [br(x) - \varepsilon, br(x) + \varepsilon]$ for every type x. We therefore, have the following corollary of Lemmas 3 and 4.

Corollary 3 *For every $\varepsilon > 0$, for large enough n every player i bids in $[br(x_i^n) - \varepsilon, br(x_i^n) + \varepsilon]$ with probability* 1*.*

To prove part (b) of the theorem it remains to show that $T \circ br$ is the assortative allocation. This is done by the following lemma.

Lemma 5 $G^{-1}(F(x)) = T(br(x))$ *for any type x.*

Thus, the mechanism that prescribes for type x prize $T(br(x))$ and bid $br(x)$ implements the assortative allocation. Moreover, the utility of type 0 is 0, because $T(br(0)) = 0$ and type 0 can get prize 0 by bidding 0, so $br(0) = 0$. Together with the observation that every type is at least as well off bidding optimally as getting utility 0 by bidding 0 and obtaining prize $T(0) = 0$, we have that the mechanism is IR.

To complete the proof, it remains to show part (a) in the statement of the theorem.[21] Let $t = br(x)$ and $T(t) = y$ for some type x, and for some $\varepsilon > 0$ let t' and t'' be such that $T(t') = y - \varepsilon/2$ and $T(t'') = y + \varepsilon/2$. Finally, let x' and x'' be such that $t' = br(x')$ and $t'' = br(x'')$. (Take $x' = 0$ and $t' = 0$ if $y - \varepsilon/2 < 0$, and $x'' = 1$ and $t'' = br(x'')$ if $y + \varepsilon/2 > 1$.)

By Lemma 3, for sufficiently large n, every player with type x' or lower bids less than the player with type x, and every player with type x'' or higher bids more than the player

[21] This part would follow easily from part (b) if we made the slightly weaker claim that every type x_i^n obtains a prize that is close to $G^{-1}(F(x_i^n))$ with high probability, but not necessarily with probability 1. Indeed, by Lemmas 3 and 5 and the continuity of T, in sufficiently large contests every type x_i^n bids with probability 1 a t such that $T(t)$ is arbitrarily close to $G^{-1}(F(x_i^n))$. By Lemma 2, this is also true if we replace $T(t)$ with $T^n(t)$. It therefore follows from (7) and the continuity of G^{-1} that every type x_i^n obtains a prize that is arbitrarily close to $G^{-1}(F(x_i^n))$ with arbitrarily high probability.

with type x. This means that the player with type x outbids at least a fraction $F(x')$ of the players, so she obtains a prize no lower than $G^{-1}(F(x')) = T(br(x')) = y - \varepsilon/2$, and outbids no more than a fraction $F(x'')$ of the players, so she obtains a prize no higher than $G^{-1}(F(x'')) = T(br(x'')) = y + \varepsilon/2$. (These inequalities are immediate if $y - \varepsilon/2 < 0$ or if $y + \varepsilon/2 > 1$.) Moreover, n can be chosen to apply simultaneously to all y (and corresponding x, x', and x''), by uniform continuity of T, which proves part (a) in the statement of the theorem.

5.2 Proof of Theorem 3

For expositional simplicity, we prove the theorem assuming that prizes have full support. Relaxing this assumption is straightforward given the proof of Theorem 2 in Appendix B.[22] The proof coincides with that of Theorem 1 up to (but not including) showing that best response sets BR_x are singletons. Without strict single crossing, best response sets may not be singletons, so we cannot pin down each player's limiting bidding behavior or a unique mechanism induced by T. Instead, we approximate the equilibria of large contests by their limits in weak*-topology.

To do so, from the subsequence of contests that corresponds to the subsequence T^n that converges uniformly to T, choose a subsequence such that D^n converges to some probability distribution D in weak*-topology. We will show that D assigns probability 1 to the set $C = \{(x, y, t) : t \in BR_x \text{ and } y = T(t)\} \subset X \times Y \times B$.

By standard arguments the correspondence that assigns BR_x to type x is upper hemi-continuous. Therefore, the set $\{(x, t) : t \in BR_x\} \subset X \times B$ is closed, and by continuity of T, C is also closed. Suppose to the contrary that D assigns a positive probability to the complement of C. Then, for some $\varepsilon > 0$, D assigns a positive probability to the complement of the 2ε-neighborhood O of C, that is, to the set $X \times Y \times B - O$. Consider the ε-neighborhood V of C and its closure $\bar{V}$ (which is contained in O), and take a continuous function f :

[22] To see why, replace T with T^*, note C is still closed, and use Lemma 12 to show (8). The rest of the proof is unchanged.

$X \times Y \times B \to [0,1]$ such that $f(\bar{V}) = 1$ and $f(X \times Y \times B - O) = 0$. Then,

$$\int f dD < 1.$$

But by Lemma 3, convergence of T^n to T, and (7), for sufficiently large n every player i with arbitrarily high probability bids t and obtains y such that $(x_i^n, y, t) \in V$. Thus,

$$\int f dD^n \to 1, \tag{8}$$

a contradiction.

Thus, the limit mechanism prescribes for each of a measure 1 of types x bids $t \in BR_x$ and corresponding prizes $T(t)$ with probability 1. This implies that D determines a tariff mechanism.[23] It is regular, because each D^n is regular. It remains to show that this mechanism implements a consistent allocation, but this follows from the fact that D^n implement consistent allocations. We show this for the marginal with respect to x; the proof for the marginal with respect to y is analogous.

Consider a continuous function $f_\varepsilon : X \times Y \times B \to [0,1]$ whose value is 1 on the set of all (x, y, t) such that $x \leq x^*$ and 0 on the set of all (x, y, t) such that $x \geq x^* + \varepsilon$, for some $\varepsilon > 0$. Then, by the definition of weak*-convergence,

$$\int f_\varepsilon dD^n \to \int f_\varepsilon dD.$$

For large enough n the integrals on the left-hand side belong to $[F(x^*) - \varepsilon, F(x^* + \varepsilon) + \varepsilon]$. These integrals would belong to $[F(x^*), F(x^* + \varepsilon)]$ if the marginals of D^n with respect to x were equal to F. But the marginal of D^n with respect to x is not equal to F; rather, it is a measure that assigns probability $1/n$ to every x_i^n. These measures, however, converge in weak*-topology to F, which implies that the integrals belong to $[F(x^*) - \varepsilon, F(x^* + \varepsilon) + \varepsilon]$. Therefore, $\int f_\varepsilon dD$ also belongs to $[F(x^*) - \varepsilon, F(x^* + \varepsilon) + \varepsilon]$. Taking the limit for $\varepsilon \to 0$, and applying right-continuity of CDFs, we obtain that F is the marginal of D with respect to x.

[23] If needed, change D on a measure 0 set of types x so that it assigns to every type x bids $t \in BR_x$ and corresponding prizes $T(t)$.

This completes the proof of the first part of the theorem. To show the second part, recall that the limit mechanism prescribes for each type x only bids $t \in BR_x$ (together with prizes $T(t)$), i.e., such bids are prescribed with probability 1.

If for some $x' < x''$ and $t' > t''$ type x' is prescribed t' and type x'' is prescribed t'', then x' weakly prefers t' to t'' and x'' weakly prefers t'' to t'. By weak single crossing, x'' weakly prefers t' to t'', and x' weakly prefers t'' to t'. That is, each of the two types is indifferent between the two bids. Thus, if a lower type is prescribed a higher prize, this is without loss of efficiency.

6 Conclusion

In large contests, players' strategies and the resulting distribution of prizes are approximated by certain single-agent mechanisms. Under strict single crossing and quasi-linearity of players' utilities, the approximating mechanism is unique. Thus, the outcomes of large contests can be approximated by using standard techniques from the mechanism design literature, even when solving for equilibrium is difficult or impossible. An immediate implication is that the outcome of large contests that satisfy strict single crossing is approximately assortative, regardless of the details of players' utilities and the distribution of prizes. Another implication is that contest design and certain comparative statics exercises can be conducted easily for large contests. For example, suppose that a designer has a budget that can be used to generate a distribution of prizes. For each distribution there is a corresponding assortative allocation, which generates some surplus, and a corresponding distribution of bids, which are unique under the aforementioned conditions. Given a welfare measure that depends on the surplus and bid distribution, such as a weighted average of the surplus and the aggregate bids, the designer can choose the prize distribution that maximizes welfare. If the designer controls the distribution of players, then similar exercises can also be performed with respect to this distribution.

The approach and results can also be used to provide foundations for contest models with a continuum of competitors, in the spirit of Morgan, Sisak, and Várdy (2013). It may also be possible to derive additional results that are weaker than Theorems 1 and 2, but stronger than Theorem 3, by identifying suitable conditions weaker than strict single crossing.

The approximation approach may also prove useful for studying other contest specifications. One example is large contests with ex-ante symmetric players, in which each player has private information about her utility function. Another example is contests in which some players have initial advantages that must be overcome by other players. Appendix C contains an example of such a contest and its approximation.[24] More generally, the approach may be useful in analyzing other large discontinuous games, such as large auctions and double auctions.

[24] Obtaining a general result for such contests is not straightforward, because they require allowing the agent's utility function to weakly decrease in his bid, whereas many of our arguments rely on the assumption that a higher bid is inferior if it does not increase the probability of obtaining better prizes.

A Proofs of results from Section 5.1

A.1 Proof of Lemma 1

Suppose the contrary for some $t \in (0, b_{\max})$, $q^m \uparrow t$, and $r^m \downarrow t$. Let $y' = \lim T(q^m)$ and $y'' = \lim T(r^m)$ (the limits exist by the monotonicity of T), and let $\gamma = (y'' - y')/4$. In what follows, the index N is assumed large enough so that $t + 1/N \leq b_{\max}$ and $t - 1/N \geq 0$.

Suppose first that $U(0, y, t)$ strictly increases in y. By uniform continuity of U and the inequality $y'' - \gamma > y' + \gamma$, there exist a $\Delta > 0$ and N such that for any $n \geq N$,

$$\min_x \left[U\left(x, y'' - \gamma, t + \frac{1}{n}\right) - U\left(x, y' + \gamma, t - \frac{1}{n}\right) \right] \geq \Delta. \tag{9}$$

Choose an element t' of the sequence q^m such that $t' \in (t - 1/N, t)$, and choose an element t'' of the sequence r^m such that $t'' \in (t, t + 1/N)$. Take a $K > 0$ small enough so that $(1 - K)\Delta - KU_{\max} > 0$, where $U_{\max} = \max_x U(x, 1, 0)$. Choose $n \geq N$ large enough so that:

Proof. $|T^n(t'') - T(t'')| < \gamma/3$ and $|T^n(t') - T(t')| < \gamma/3$.

For all i, $\Pr((A^n(t'') - R_i^n(t'')) > \alpha) < K$ and $\Pr((R_i^n(t') - A^n(t')) > \alpha) < K$, (see (7)) where α is small enough so that $T^n(t'') - G^{-1}(A^n(t'') - \alpha) < \gamma/3$ and $G^{-1}(A^n(t') + \alpha) - T^n(t') < \gamma/3$ (recall that G^{-1} is continuous). ■

Then, in the equilibrium that corresponds to T^n, no player bids in $[t - 1/N, t']$ with positive probability, because such bids give lower payoff than bidding slightly above t''. To see why, note that the payoff of any player i from bidding in $[t - 1/N, t']$ is at most

$$(1 - K)\left(U\left(x_i^n, T(t') + \frac{2\gamma}{3}, t - \frac{1}{N}\right)\right) + KU_{\max} \leq (1 - K)\left(U\left(x_i^n, y' + \gamma, t - \frac{1}{N}\right)\right) + KU_{\max},$$

and the payoff from bidding slightly above t'' is at least

$$(1 - K)\left(U\left(x_i^n, T(t'') - \frac{2\gamma}{3}, t + \frac{1}{N}\right)\right) \geq (1 - K)\left(U\left(x_i^n, y'' - \gamma, t + \frac{1}{N}\right)\right).$$

The difference in the two payoffs is therefore at least

$$(1 - K)\left(U\left(\frac{i}{n}, y'' - \gamma, t + \frac{1}{N}\right) - U\left(\frac{i}{n}, y' + \gamma, t - \frac{1}{N}\right)\right) - KU_{\max} \geq (1 - K)\Delta - KU_{\max},$$

which is positive by the definition of K.

This shows that no player bids in $[t - 1/N, t']$ with positive probability, so by the No-Gap Property $T^n(t') = 1$. But $T^n(t') \to T(t') \leq y' < y'' \leq 1$, a contradiction.

The difficulty when $U(0, y, t)$ weakly increases in y is that there may not be a positive Δ for which (9) holds. Nevertheless, for any $\varepsilon > 0$, the arguments above hold for all x in $[\varepsilon, 1]$, and in particular there exist rational t' and t'' such that for large n (1) players with types higher than ε prefer to increase their bid by $t'' - t'$ in order to improve their prize by $\gamma/2$ and (2) $T^n(t'') - T^n(t') > \gamma$. Letting $\hat{t} = \inf\left\{\tilde{t} : T^n\left(\tilde{t}\right) \geq T^n(t'') - \gamma/2\right\} \in (t', t'')$, we see that only players with types no higher than ε can bid in $\left[t', \hat{t}\right]$. But for sufficiently small ε (and using uniform continuity of G^{-1}) this would not be enough to increase $T^n(t')$ to $T^n(t') + \gamma/2$, contradicting $T^n\left(\hat{t}\right) \geq T^n(t'') - \gamma/2$ (recall that A^n is right continuous).

For the case $t = b_{\max}$, set $t'' = b_{\max}$ and repeat the argument above.[25]

Suppose now that $t = 0$. Then the above proof, with $t' = 0$ instead of $t' = t - 1/2N$, shows that for large n no player bids $t' = 0$ with positive probability. This means, in turn, that sufficiently small bids give lower payoff than the bid t''. Thus, no player bids close to $t' = 0$ with positive probability, which contradicts the No-Gap Property.

A.2 Proof of Lemma 2

Suppose the contrary. Then there is some $\delta > 0$ and a sequence of integers $n_1, n_2, \ldots, n_k, \ldots$ such that for every n_k there is some bid t_k with $|T^{n_k}(t_k) - T(t_k)| > \delta$. Passing to a subsequence if necessary, we assume that the sequence $t_k \to t$.

Consider rationals q' and q'' such that $q' < t < q''$ and $T(q'') - T(q') < \delta/2$; such numbers exist because T is continuous.[26] For large enough values of k, we have that $|T^{n_k}(q') - T(q')| < \delta/2$ and $|T^{n_k}(q'') - T(q'')| < \delta/2$.

For any $t' \in [q', q'']$, either (a) $T^{n_k}(t') \geq T(t')$, or (b) $T^{n_k}(t') \leq T(t')$.

By the monotonicity of T and T^{n_k}, we have

$$T^{n_k}(t') - T(t') \leq T^{n_k}(q'') - T(q') \leq |T^{n_k}(q'') - T(q'')| + |T(q'') - T(q')| < \delta$$

in case (a), and

$$T(t') - T^{n_k}(t') \leq T(q'') - T^{n_k}(q') \leq |T(q'') - T(q')| + |T(q') - T^{n_k}(q')| < \delta$$

in case (b).

[25] The only difference is that bidding "slightly above $b_{\max}$" is impossible. But by bidding $b_{\max}$ a player wins with probability 1, because $b_{\max}$ is strictly dominated by 0 for all players.

[26] If $t = 0$ set $q' = 0$, and if $t = b_{\max}$ set $q'' = b_{\max}$.

Since $t_k \in [q', q'']$ for large enough values of k, we obtain a contradiction to the assumption that $|T^{n_k}(t_k) - T(t_k)| > \delta$ for all such k.

A.3 Proof of Lemma 3

Suppose to the contrary that for arbitrarily large n, the strategy of some player i in the n-th contest assigns a positive probability to the complement of $BR_{x_i^n}(\varepsilon)$. Passing to a convergent subsequence if necessary, we assume that $x_i^n \to x^*$.

Note that for every x there is a $\delta_x > 0$ such that any bid from the complement of $BR_x(\varepsilon)$ gives type x a payoff lower than any element of BR_x by at least δ_x. Otherwise, by taking a suitable subsequence, we would show that there exists an element of BR_x that belongs to the complement of $BR_x(\varepsilon)$. Let $\delta = \delta_{x^*}$.

Observe that:

1. The maximal payoff of type x, attained at any bid from BR_x, is continuous in x.

Indeed, by continuity of the payoff function, any bid $t \in BR_x$ yields to any type close enough to x, a payoff close to that obtained by type x by bidding t. Thus, the maximal payoff function is lower semi-continuous.[27]

Suppose now that the function is not upper semi-continuous. That is, there is a $\rho > 0$ and a sequence of types $x_k \to x$ such that by bidding some t_k type x_k obtains a payoff at least ρ higher than the maximal payoff of type x. Passing to a convergent subsequence if necessary, we assume that $t_k \to t$. By continuity of the payoff functions, by bidding t type x obtains a payoff higher by at least $\rho > 0$ than her maximal payoff, a contradiction.

2. For every $\rho > 0$, for sufficiently large n the highest payoff that type x_i^n can obtain by bidding in the complement of $BR_{x_i^n}(\varepsilon)$ is higher by at most ρ than the highest payoff that type x^* can obtain by bidding in the complement of $BR_{x^*}(\varepsilon)$.

Indeed, suppose that for $n_k > k$ type $x_i^{n_k}$ obtains by bidding some t_k in the complement of $BR_{x_i^{n_k}}(\varepsilon)$ a payoff at least ρ higher than the highest payoff that type x^* can obtain by bidding in the complement of $BR_{x^*}(\varepsilon)$. Passing to a convergent subsequence if necessary, we assume that $t_k \to t$. Since every $(x_i^{n_k}, t_k)$ belongs to the complement of $BR(\varepsilon)$, so does (x^*, t); thus, (x^*, t) belongs to the complement of $BR_{x^*}(\varepsilon)$. However, by continuity of the

[27] Lower semi-continuity says that the value of the limit is no lower than the limit of values, while upper semi-continuity says that the value of the limit is no higher than the limit of values. A function is continuous if and only if it is upper and lower semi-continuous.

payoff functions, bidding t gives type x^* a payoff by at least ρ higher than the highest payoff that type x^* can obtain by bidding in the complement of $BR_{x^*}(\varepsilon)$, a contradiction.

By 1 and 2, for sufficiently large n, any bid in the complement of $BR_{x_i^n}(\varepsilon)$ gives type x_i^n a payoff lower by at least $\delta/2$ than any bid in $BR_{x_i^n}$. By uniform convergence of T^n to T, the analogous statement, with $\delta/2$ replaced with some smaller positive number and T replaced with T^n, is also true. This means, however, that for sufficiently large n, player i would be strictly better off bidding slightly above any bid in $BR_{x_i^n}$ than bidding in the complement of of $BR_{x_i^n}(\varepsilon)$. This is because (7) implies that for sufficiently large n, by bidding slightly above t the player obtains a prize arbitrarily close to $T^n(t)$ with probability arbitrarily close to 1.

A.4 Proof of Lemma 4

Observe that for any $x' < x''$, strict single crossing implies that if $t' \in BR_{x'}$ and $t'' \in BR_{x''}$, then $t' \leq t''$. Suppose that $BR_{x'}$ contained two bids, $t_1 < t_2$, for some type x'. The first observation and Lemma 3 imply that for any $0 < \varepsilon < (t_2 - t_1)/3$, for sufficiently large n only players with types in $I = [\max\{x' - \varepsilon, 0\}, \min\{x' + \varepsilon, 1\}]$ may bid in the interval $[t_1 + (t_2 - t_1)/3, t_2 - (t_2 - t_1)/3]$.

Proof. Let

$$\max_{x \in X, y \in Y}[U(x, G^{-1}(\max\{G(y) + \Delta, 1\}), t_2 - \frac{t_2 - t_1)}{2}) - U(x, y, t_1 + \frac{(t_2 - t_1}{3})]$$

be the maximal possible gain in utility when bidding $t_2 - (t_2 - t_1)/2$ instead of $t_1 + (t_2 - t_1)/3$ and getting a prize that is Δ higher in the distribution of prizes. This gain becomes negative, and bounded away from 0, for sufficiently small $\Delta > 0$. Indeed, suppose that for a sequence $\Delta_n \to 0$ there is a sequence (x_n, y_n) such that

$$\liminf[U(x_n, G^{-1}(\max\{G(y_n) + \Delta_n, 1\}), t_2 - \frac{t_2 - t_1}{2}) - U(x_n, y_n, t_1 + \frac{t_2 - t_1}{3})] \geq 0.$$

Passing to a subsequence if necessary, consider the limit (x^*, y^*) of (x_n, y_n). Observe that by upper semi-continuity of G we have that $\limsup G(y^n) + \Delta_n \leq G(y^*)$. By monotonicity and continuity of G^{-1}, this last inequality implies that $\limsup G^{-1}(G(y^n) + \Delta_n) \leq G^{-1}(G(y^*)) = y^*$. Therefore, by continuity of U we would have that

$$U(x^*, y^*, t_2 - \frac{t_2 - t_1}{2}) - U(x^*, y^*, t_1 + \frac{t_2 - t_1}{3}) \geq 0,$$

a contradiction.

Taking $\Delta = F(\min\{x'+\varepsilon,1\}) - F(\max\{x'-\varepsilon,0\})$, for sufficiently small $\varepsilon > 0$, we obtain that for sufficiently large n every player is better off bidding $t_1 + (t_2 - t_1)/3$ than bidding $t_2 - (t_2 - t_1)/2$. Moreover, every player is better off bidding $t_1 + (t_2 - t_1)/3$ than bidding any bid in $[t_2 - (t_2 - t_1)/2, t_2 - (t_2 - t_1)/3]$, because such bids are even more costly than $t_2 - (t_2 - t_1)/2$, whereas the fraction of additional players defeated is still at most Δ. Therefore, no player bids in the interval $[t_2 - (t_2 - t_1)/2, t_2 - (t_2 - t_1)/3]$. But then, by the No-Gap Property, no player bids more than $t_2 - (t_2 - t_1)/3$. This implies that $T^n(t_2 - (t_2 - t_1)/3)$, so $T(t_2 - (t_2 - t_1)/3)$ is equal to 1. Therefore, t_2 is not in $BR_{x'}$, because bidding slightly above $t_2 - (t_2 - t_1)/3$ gives type x' a higher payoff.

Consequently, BR_x is a singleton for any x, and by strict single crossing the corresponding function br is weakly increasing. An argument analogous to the argument that showed the singleton property shows that br is continuous. ■

A.5 Proof of Lemma 5

Consider first some type x such that $x^{\min} = \min\{z : br(z) = br(x)\} > 0$ and $x^{\max} = \max\{z : br(z) = br(x)\} < 1$ ($x^{\min}$ and $x^{\max}$ are well defined because br is continuous). Take any $\delta \in (0, \min\{x^{\min}, 1 - x^{\max}\})$. By Corollary 3 applied to $\varepsilon = (br(x^{\min}) - br(x^{\min} - \delta))/2$, if n is sufficiently large, then the equilibrium bids of every player with type lower than $x^{\min} - \delta$ are lower than $br(x^{\min} - \delta) + \varepsilon$. Therefore, a player who bids $br(x^{\min})$ outbids all players with types lower than $x^{\min} - \delta$ and obtains a prize $y \geq G^{-1}(F(x^{\min} - \delta))$. Consequently, $T(br(x^{\min})) \geq G^{-1}(F(x^{\min} - \delta))$, and because F and G^{-1} are continuous, by taking δ to 0 we obtain $T(br(x^{\min})) \geq G^{-1}(F(x^{\min}))$. Similar arguments show that $T(br(x^{\min} - \delta)) \leq G^{-1}(F(x^{\min}))$, and because T and br are continuous, by taking δ to 0 we obtain $T(br(x^{\min})) \leq G^{-1}(F(x^{\min}))$. Similarly, for sufficiently large n, the bids of all players with types higher than $x^{\max} + \delta$ are higher and bounded away from $br(x^{\max})$, so $T(br(x^{\max})) \leq G^{-1}(F(x^{\max} + \delta))$, and by taking δ to 0 we obtain $T(br(x^{\max})) \leq G^{-1}(F(x^{\max}))$; and similar arguments show that $T(br(x^{\max})) \geq G^{-1}(F(x^{\max}))$. Since $br(x) = br(x^{\min}) = br(x^{\max})$, this yields that $T(br(x)) = G^{-1}(F(x^{\min})) = G^{-1}(F(x^{\max}))$. Therefore, $T(br(x)) = G^{-1}(F(x))$ (because $G^{-1} \circ F$ is monotonic).

Suppose that $x^{\min} > 0$ and $x^{\max} = 1$. Then, $T(br(x^{\min})) = G^{-1}(F(x^{\min}))$ by an analogous argument to the previous case. And because $T(br(x^{\min})) = T(br(1))$ and $G^{-1} \circ F$

is weakly increasing, we have $T(br(z)) \le G^{-1}(F(z))$ for any $z \in \left[x^{\min}, 1\right]$. To show the reverse inequality, it suffices to show that $T(br(1)) = 1$, because $G^{-1}(F(z)) \le 1$ for any z. Suppose to the contrary that $T(br(1)) = y < 1$. This means that for sufficiently large n by bidding $br(1)$ a player obtains a prize bounded away from 1 with arbitrarily high probability. On the other hand, for any $\varepsilon > 0$, if n is sufficiently large, then by bidding $br(1)+\varepsilon$ a player obtains prize 1 with probability 1. Thus, if ε is sufficiently small and n is sufficiently large, then no player bids in an interval slightly below $br(1)$, contradicting the No-Gap Property.

Now suppose that $x^{\min} = 0$ and $x^{\max} < 1$. Then, the arguments above show that $T(br(x^{\max})) = G^{-1}(F(x^{\max}))$. And because $T(br(0)) = T(br(x^{\max}))$ and $G^{-1} \circ F$ is weakly increasing, we have $T(br(z)) \ge G^{-1}(F(z))$ for any $z \in [0, x^{\max}]$. To show the reverse inequality, it suffices to show that $T(br(0)) = 0$, because $G^{-1}(F(z)) \ge 0$ for any z. Because $T(0) = 0$, it suffices to show that $br(0) = 0$. Suppose to the contrary that $br(0) = t > 0$.

The proof of Lemma 3 establishes that all the best responses of a player with type x_i^n (to the equilibrium strategies of the other players), and not only her equilibrium bids, belong to $BR_{x_i^n}(\varepsilon)$. Thus, by the monotonicity of br and this stronger result, for any small $\varepsilon > 0$ and large enough n no player has best responses lower than $t - \varepsilon > 0$.

For such an n, denote by $t^* > 0$ the lowest bid such that some player bids arbitrarily close to t^* with positive probability. Suppose that some player i has an atom at t^*. This player obtains the lowest prize by bidding t^*. Indeed, since the bids of all other players are no lower than t^* with probability 1, by bidding t^* player i can obtain a prize higher than the lowest one only as a result of favorable tie breaking at t^*. But then player i would be better of by bidding slightly above t^* instead of bidding t^*. On the other hand, any positive bid that gives the lowest prize cannot be any players's best response (she could get the lowest prize or better by bidding 0). Therefore, no player has an atom at t^*. By the definition of t^*, there is a player who makes bids arbitrarily close to t^* with positive probability. Because no player has an atom at t^*, the probability that the player obtains prize 0 by making bids that approach t^* approaches 1. Therefore, such bids cannot be the player's best response either (she would be better off bidding 0).

Finally, it cannot be that $x^{\min} = 0$ and $x^{\max} = 1$. This is because by the previous parts of the lemma, $x^{\max} = 1$ implies $T(br(1)) = 1$ and $x^{\max} = 0$ implies $T(br(0)) = 0$, which contradicts the definitions of $x^{\min}$ and $x^{\max}$.

B Proof of Theorem 2

Recall that $G^{-1}(z) = \inf\{y : G(y) \geq z\}$, and note that G^{-1} may be discontinuous (but is left-continuous). This requires modifying almost all the arguments used in the proof of Theorem 1. In order not to obscure the structure of the proof, we relegate to the end of the section the proofs of all intermediate results.

Let $I_0 = (y_0^l, y_0^u)$ be a longest interval in $[0, G^{-1}(1)]$ to which G assigns measure 0; let $I_1 = (y_1^l, y_1^u)$ be a longest such interval disjoint from I_1, and so on. Then, every open interval of prizes that has measure zero is contained in one of the intervals $I_0, I_1, \ldots$. And for any $\varepsilon > 0$, there is a K such that the lengths of I_{K+1}, $I_{K+2}, \ldots$ sum up to less than ε.

The definitions of R_i^n, A_i^n, A^n, and T^n are as in Section 5.1. The definition of T, however, must be changed. We first define a function A on the rationals in B. Take an ordering of all rationals in B, denoted by $q_1, q_2, \ldots$. Take a converging subsequence of the sequence $A^n(q_1)$, denote it by $A^{n_1}(q_1)$, and denote its limit by $A(q_1)$. Take a converging subsequence of the sequence $A^{n_1}(q_2)$, denote it by $A^{n_2}(q_2)$, and denote its limit by $A(q_2)$. Continue in this fashion to obtain a function $A : \{q_1, q_2, \ldots\} \to [0, 1]$, which is weakly increasing (because each A^n is). In addition, define a subsequence of A^n such that its k-th element is the k-th element in the sequence A^{n_k}. For the rest of the proof, denote this new sequence by A^n (with the corresponding sequence $T^n = G^{-1} \circ A^n$).

Let $T = G^{-1} \circ A$. We now list some properties of T:

(1) $T \leq \liminf T^n$ (because G^{-1} is left-continuous), and $T = \lim T^n$ if G^{-1} is continuous.

(2) T is (weakly) increasing, because G^{-1} and A are.

(3) $T(b_{\max}) = G^{-1}(1)$, because $A^n(b_{\max}) = 1$ so $A(b_{\max}) = 1$.

The discontinuities in G^{-1} imply that T defined on the rationals in B may not be continuous, so Lemma 1 does not hold. Points of discontinuity, however, correspond to open intervals of prizes that have measure zero. More precisely, we have the following result.

Lemma 6 *For any $t > 0$ in B (not necessarily rational) one of the following conditions holds:*

1. *For any two sequences $q^m \uparrow t$ and $r^m \downarrow t$ of rationals in B, we have $\lim T(q^m) = \lim T(r^m)$.*

2. *There is some $k = 1, 2, \ldots$ such that for any two sequences $q^m \uparrow t$ and $r^m \downarrow t$, of rationals*

in B, we have $\lim T(q^m) = y_k^l$ *and* $\lim T(r^m) = y_k^u$. *Moreover,* $\lim A(q^m) = G(y_k^l)$ *and* $\lim A(r^m) = G(y_k^u)$.

Using Lemma 6, we define a function T^* on the entire interval B by setting $T^*(t) = \lim T(r^m)$ for some sequence $r^m \downarrow t$ of rationals in B (and $T^*(b_{\max}) = G^{-1}(1)$). Monotonicity of T on the rationals guarantees that $T^*(t)$ is well-defined, and is the same regardless of the sequence r^m. In addition, it is easy to check that T^* is (weakly) monotonic, right-continuous, and continuous at every bid t such that condition 1 from Lemma 6 holds. Note that T^* may not be an extension of T, because when $\lim T(r^m) \neq T(t)$ for a rational t, we have that $T^*(t) = \lim T(r^m) \neq T(t)$.

Consider now a bid $t > 0$ such that condition 2 from Lemma 6 holds. Denote this bid t by t_k, where k is described in condition 2. Then, there is a bid $t' < t_k$ such that $A(t') = A(t) = G(y_k^l)$, so A is constant on an interval below t_k. Indeed, if $A(t') < G(y_k^l)$ for all $t' < t_k$, then, as in the proof of Lemma 6, for large n no player would bid any t' slightly below t_k, because bidding slightly above t_k would almost certainly give a prize no lower than y_k^u, whereas bidding t' would almost certainly give a prize no higher than y_k^l. Let

$$t_k^l = \inf\left\{t' : A(t') = G\left(y_k^l\right)\right\} < t_k.$$

It is also true that every maximal interval on which T^* is constant with a value lower than $G^{-1}(1)$ is $\left[t_k^l, t_k\right)$ for some k. Indeed, consider a maximal nontrivial interval with lower bound t^l and upper bound t^u on which the value of T^* is $y < G^{-1}(1)$. It suffices to show that $T^*(t^u) > y$, because then condition 2 from Lemma 6 applies to t^u, which implies that $t^u = t_k$ for some k; and the maximality of $\left[t^l, t^u\right)$ yields $t^l = t_k^l$. Suppose that $T^*(t^u) = y$. Then, for large enough n bidding t^u almost certainly gives a prize at most slightly higher than y, whereas bidding slightly above t^l almost certainly gives a prize not much lower than y. The former statement follows directly from right-continuity T^* and the assumption that $T^*(t^u) = y$. The latter statement is obvious for $t^l = 0$. And for $t^l > 0$ it follows from the observation that either T^* is continuous at t^l, in which case the statement is obvious, or T^* is discontinuous at t^l, in which case G has an atom at $T^*\left(t^l\right) = y$. But then, for large enough n, no player bids in some neighborhood of t^u, because bidding slightly above t^l leads to a higher payoff. This contradicts the No-Gap Property, because $y < G^{-1}(1)$.

Because G^{-1} may be discontinuous, T^n need not uniformly converge to T or T^*, even on the set of points at which they are continuous. In particular, for a rational t in $\left[t_k^l, t_k\right)$

it may be that $A^n(t) > A(t) = G\left(y_k^l\right)$ for arbitrarily large n, so $T^n(t) = G^{-1}(A^n(t)) \geq y_k^u > y_k^l$, whereas $T^*(t) = y_k^l$. Nevertheless, T^n "almost converges" uniformly except on some neighborhoods of a finite number of intervals $[t_k^l, t_k]$. More precisely, we say that T^n *converges uniformly to* T^* *up to* β on a set C if there exists an N such that for every $n \geq N$ and $t \in C$ we have that

$$|T^n(t) - T^*(t)| < \beta.$$

We then have the following modification of Lemma 2.

Lemma 7 *For every $\beta > 0$, there exists a number K such that for every $\gamma > 0$, T^n converges uniformly to T^* up to β on the complement of*

$$O_\gamma = \bigcup_{k=1}^{K} (t_k^l - \gamma, t_k + \gamma).$$

We now relate players' equilibrium behavior in large contests to the inverse tariff T^*. Define BR_x, $BR(\varepsilon)$ and $BR_x(\varepsilon)$ as in Section 5.1 with T^* instead of T (the maximal payoff is achieved because T^* is increasing and right-continuous, so is upper semi-continuous). Define *the mass expended in an interval of bids I (by a set of players M in the n-th contest)* as $\left(\sum_{i \in M} \Pr\{\sigma_i^n \in I\}\right)/n$. We then have the following result, which we use in proving the remaining results.

Lemma 8 *For any interval $\left[t_k^l, t_k\right]$ and any $\varepsilon > 0$ and $L > 0$, there exists $\gamma > 0$ such that for sufficiently large n we have that:*

(i) The mass expended in $[t_k^l - \gamma, t_k^l + \gamma)$ by players with types x for which $t_k^l \notin BR_x(\varepsilon)$ is less than $\varepsilon/3L$;

(ii) The mass expended in $(t_k - \gamma, t_k]$ by players with types x for which $t_k \notin BR_x(\varepsilon)$ is less than $\varepsilon/3L$.

In addition, for any $\alpha > 0$, for sufficiently large n we have that:

(iii) The mass expended in $\left[t_k^l + \alpha, t_k - \alpha\right]$ by all players is less than $\varepsilon/3L$.

Lemma 3 must also to be modified.

Lemma 9 *For every $\varepsilon > 0$, there exist K such that for every $\gamma > 0$ and sufficiently large n,*[28] *the equilibrium bid of every player i in the n-th contest belongs with probability 1 to*

$$BR_{x_i^n}(\varepsilon) \cup \bigcup_{k=1}^{K} (t_k^l - \gamma, t_k).$$

Strict single crossing no longer implies that BR_x is a singleton. Instead, we have the following result.

Lemma 10 *If strict single crossing holds, then for all but a countable number of types the set BR_x is a singleton. For those types for which it is not a singleton, BR_x contains precisely two elements: t_k^l and t_k for some k. The correspondence that assigns to type x the set BR_x is weakly increasing (i.e., for any $x' < x''$, if $t' \in BR_{x'}$ and $t'' \in BR_{x''}$, then $t' \leq t''$) and upper hemi-continuous.*

Let $br(x) = \min BR_x$, and note that br is increasing and left continuous, and is not right continuous precisely at types x for which BR_x is not a singleton. We then have the following corollary of Lemmas 8, 9, and 10, which is a modification of Corollary 3.

Corollary 4 *For every $\varepsilon > 0$, for large enough n a fraction $1 - \varepsilon$ of the players i bid in $[br(x_i^n) - \varepsilon, br(x_i^n) + \varepsilon]$ with probability $1 - \varepsilon$.*

To prove part (b) of the theorem it remains to show that $T^* \circ br$ is the assortative allocation. This is done by the following lemma, which is a modification of Lemma 5 that accommodates the discontinuities in T^* and br.

Lemma 11 $G^{-1}(F(x)) = T^*(br(x))$ *for any type $x > 0$.*

To complete the proof, it remains to show (a) in the statement of the theorem. To do so, we use the following result, which we also use to prove that Theorem 3.

Lemma 12 *For every $\varepsilon, \delta > 0$, and sufficiently large n, there exists an event $E_n \subset X \times Y \times B$ such that $D^n(E_n) \geq 1 - \varepsilon$ and for every (x_i^n, y, t) in E_n we have that:*

(1) $|y - T^*(t)| < \delta$, or

(2) $|t - r| < \delta$ and $|y - T^*(r)| < \delta$ for some r in $BR_{x_i^n}$.

If strict single crossing holds, then (2) holds for every (x_i^n, y, t) in E_n.

[28] How large n needs to be may depend on γ.

To see that Lemma 12 implies (a) in the statement of the theorem, choose some $\varepsilon > 0$. Lemma 12 with $\varepsilon^2/2$ instead of ε, shows that for every $\delta > 0$, there is a large enough n_δ such that for a fraction $1-\varepsilon/2$ of the players i, the conditional probability of $E^n \cap \{x_i^n\} \times Y \times B$ is at least $(1-\varepsilon)/n$, i.e.,each such player i obtains with probability at least $1-\varepsilon$ a prize y that differs by at most δ from the prize T^* prescribes to some optimal bid of the player's type. Let $\delta = \varepsilon/2$, denote by K the number of types x for which $T^*(\max BR_x) - T^*(\min BR_x) > \delta$, and choose n_δ large enough so that $K/n_\delta < \varepsilon/2$. Then, by excluding the players with types x for which $T^*(\max BR_x) - T^*(\min BR_x) > \delta$, we have that for large enough n each of a fraction $1-\varepsilon$ of the players i obtains with probability at least $1-\varepsilon$ a prize y such that $|y - br(x_i^n)| < \varepsilon$, because for some t in $BR_{x_i^n}$ we have $|y - br(x_i^n)| \leq |T^*(t) - br(x_i^n)| + |y - T^*(t)| \leq 2\delta = \varepsilon$.

B.1 Proof of Lemma 6

Let $\lim T(q^m) = y'$ and $\lim T(r^m) = y''$. Both limits y' and y'' exist and $y' \leq y''$ by monotonicity. Suppose that $y' < y''$. If G assigns a positive measure to (y', y''), then it assigns a positive measure to any interval with endpoints sufficiently close to y' and y''. In such a case, we obtain a contradiction by arguments similar to those used in the proof of Lemma 1. Indeed, for sufficiently large n no bidder would bid slightly below t, because bidding slightly above t would almost certainly give a better prize.

Thus, G assigns measure zero to (y', y''). This implies that $(y', y'') \subseteq (y_k^l, y_k^u)$ for some k. By definition, T takes values in $[0, y_k^l] \cup [y_k^u, 1]$, so $y' = y_k^l$ and $y'' = y_k^u$. Moreover, the monotonicity of T implies that k is the same for any sequences $q^m \uparrow t$ and $r^m \downarrow t$ of rationals in B. It remains to show that $\lim A(q^m) = G(y_k^l)$ and $\lim A(r^m) = G(y_k^u)$.

For this, note that if $\lim A(q^m) > G(y_k^l)$, then $\lim T(q^m) > y_k^l$. Similarly, if $\lim A(r^m) > G(y_k^u)$, then $\lim T(r^m) > y_k^u$. The inequalities $\lim A(q^m) < G(y_k^l)$ and $\lim A(r^m) < G(y_k^u)$ can be ruled out similarly if G does not have atoms at y_k^l or y_k^u. Suppose that G has an atom at y_k^u and $\lim A(r^m) < G(y_k^u)$. Since $\lim T(r^m) = y_k^u$, $A(r^m) > G(y_k^l)$ for sufficiently large m. Take two rationals r^m such that $G(y_k^l) < A(r^m) < G(y_k^u)$; denote them by $t' < t''$. Then, for sufficiently large n any player obtains prize y_k^u with arbitrarily high probability by bidding any $t \in [t', t'']$. Thus, for sufficiently large n, no player would bid with non-zero probability any $t \in [(t'+t'')/2, t'']$. This contradicts the No-Gap Property.

Suppose that G has an atom at y_k^l and $\lim A(q^m) < G(y_k^l)$. Then, for sufficiently large n bidding q^m almost certainly gives a prize no better than y_k^l. In contrast, bidding r^m almost certainly gives a prize at least as good as y_k^u. This follows directly from (7) if G has an atom at y_k^u. If G does not, then this again follows from (7) for large enough n, because $A\left(r^m\right) > G\left(y_k^u\right)$ for any m. For large enough n a contradiction is obtained similarly to the last part of the proof of Lemma 1.

B.2 Proof of Lemma 7

The proof is analogous to the proof of Lemma 2. Take a K such that the lengths of I_{K+1}, $I_{K+2}, \ldots$ sum up to less than $\beta/2$. Take any $\gamma > 0$, and suppose to the contrary that there is an increasing sequence of integers $n_1, n_2, \ldots, n_m, \ldots$ such that for every n_m there is some bid $t_m \notin O_\gamma$ with $\left|T^{n_m}\left(t_m\right) - T^*\left(t_m\right)\right| \geq \beta$. Passing to a subsequence if necessary, we assume that the sequence $t_m \to t$. Take rationals q' and q'' such that $q' < t < q''$ and $T^*\left(q''\right) - T^*\left(q'\right) < \beta/2$,[29] and

$$[q', q''] \subset B - \bigcup_{k=1}^{K} [t_k^l, t_k].$$

This is possible, since the lengths of I_{K+1}, $I_{K+2}, \ldots$ sum up to less than $\beta/2$. In addition, for large enough k we have that $\left|T^{n_k}\left(q'\right) - T^*\left(q'\right)\right| < \beta/2$ and $\left|T^{n_k}\left(q''\right) - T^*\left(q''\right)\right| < \beta/2$, since the length of each I_{K+1}, $I_{K+2}, \ldots$ is less than $\beta/2$. The rest of the proof coincides with the proof of Lemma 2.

B.3 Proof of Lemma 8

First, observe that the maximal payoff of type x, attained at any bid in BR_x, is still continuous in x. The proof is almost the same as that in Lemma 3. The only difference is that T^* is upper semi-continuous, as opposed to the continuous T in the proof of Lemma 3. But upper semi-continuity is all that is needed for the continuity of the maximal payoff.

This observation implies that there exists a $\delta > 0$ such that for any type x any bid in the complement of $BR_x\left(\varepsilon\right)$ gives type x a payoff lower by at least δ than any bid in BR_x. Indeed, suppose that for every $m = 1, 2, \ldots$ there exist type x_m and bid t_m in the complement of $BR_{x_m}\left(\varepsilon\right)$ that gives type x_m a payoff lower by less than $1/m$ than any bid in BR_x. Passing to a convergent subsequence if necessary, assume that $t_m \to t$ and $x_m \to x$. Bidding t gives

[29] If $t = 0$, take $q' = 0$.

type x a payoff no lower than any bid in BR_x, by upper semi-continuity of T^*. This means that $t \in BR_x$. But this cannot be, because the complement of $BR(\varepsilon)$ is a closed set that contains (x_m, t_m) for all m, so t belongs to the complement of $BR_x(\varepsilon)$.

For (i), suppose in contradiction that for any $\gamma > 0$ there are arbitrarily large n such that the mass expended in $[t_k^l - \gamma, t_k^l + \gamma)$ by players with types x for which $t_k^l \notin BR_x(\varepsilon)$ is at least $\varepsilon/3L$. Take γ small enough so that the payoff that such players obtain by bidding slightly more than any bid in BR_x is higher by $\delta/2$ than the payoff that they would obtain by bidding $t_k^l - \gamma$ and getting y_k^l.[30]

Suppose first that $t_k^l > 0$. We can assume, due to the Subsequence Property, that the sequence $A^n\left(t_k^l - \gamma\right)$ converges. Denote its limit by z. It cannot be that $z \geq G\left(y_k^l\right)$, because of the monotonicity of A^n and the definition of t_k^l.

Thus, $z < G\left(y_k^l\right)$. Take a positive $\alpha < \min\left\{\varepsilon/6L, G\left(y_k^l\right) - z\right\}$). For $t \geq t_k^l - \gamma$ and sufficiently large n, if $A^n(t) < G\left(y_k^l\right) - \alpha/2$, then no player of type x such that $t_k^l \notin BR_x(\varepsilon)$ bids t, because such a player would obtain a strictly higher payoff by bidding slightly more than any bid in BR_x. Let γ_n be defined by $t_k^l - \gamma + \gamma_n = \inf\left\{t : A^n(t) \geq G\left(y_k^l\right) - \alpha\right\}$. Since $z < G\left(y_k^l\right) - \alpha$ and $A^n(t)$ is right-continuous, we have that $\gamma_n > 0$ (for sufficiently large n). And since for every $t < t_k^l - \gamma + \gamma_n$ we have $A^n(t) < G\left(y_k^l\right) - \alpha < G\left(y_k^l\right) - \alpha/2$ (by definition of γ_n), the mass of at least $\varepsilon/3L$ must be expended in $[t_k^l - \gamma + \gamma_n, t_k^l + \gamma)$.

If a mass of at least $\varepsilon/4L$ is expended in $\left(t_k^l - \gamma + \gamma_n, t_k^l + \gamma\right)$ by these players, then we have that $A^n\left(t_k^l + \gamma\right) \geq A^n\left(t_k^l - \gamma + \gamma_n\right) + \varepsilon/4L \geq G\left(y_k^l\right) - \alpha + \varepsilon/4L > G\left(y_k^l\right) + \varepsilon/12L$. But A is monotone and its value on the rationals in $\left[t_k^l, t_k\right)$ is $G\left(y_k^l\right)$ by definition of t_k^l, a contradiction. Otherwise, the players with types x for which $t_k^l \notin BR_x(\varepsilon)$ bid precisely $t_k^l - \gamma + \gamma_n$ with probability at least $\varepsilon/12L$. Since these players tie with each other at $t_k^l - \gamma + \gamma_n$, by bidding $t_k^l - \gamma + \gamma_n$ they must obtain a prize of a specific type y with probability 1, even if they lose all ties at $t_k^l - \gamma + \gamma_n$. (Otherwise, each of them could obtain

[30] To see why bidding slightly above any $t \in BR_x$ gives a payoff close to $U(x, T^*(t), t)$, consider the following two cases:

(a) $G^{-1}(A(r^m)) > T^*(t)$ for all rationals $r^m > t$; in this case, for any $r^m > t$, if n is sufficiently large, then the player obtains a prize higher than $T^*(t)$ with arbitrarily high probability by bidding any r^m.

(b) $G^{-1}(A(r^m)) = T^*(t)$ for rationals $r^m > t$ close enough to t; in this case, $T^*(r^m) = T^*(t)$ for such rationals r^m. This implies that $t = t_{k\prime}^l$ for some $k\prime$, and for any bid $t_{k\prime}^l$, we have that G^{-1} is left-continuous at $G(T^*\left(t_{k\prime}^l\right))$.

a strictly better distribution of prizes by bidding slightly above $t_k^l - \gamma + \gamma_n$ and winning the ties at $t_k^l - \gamma + \gamma_n$.) But a player who loses all ties at $t_k^l - \gamma + \gamma_n$ has rank order no higher than $G\left(y_k^l\right) - \alpha$, by definition of γ_n, so $y \leq y_k^l$. Therefore, such a player would obtain a strictly higher payoff by bidding slightly more than any bid in BR_x.

Now suppose that $t_k^l = 0$. We can assume, due to the Subsequence Property, that the sequence $A^n\left(t_k^l - \gamma\right)$ converges. Denote its limit by z. The monotonicity of A^n and the definition of t_k^l.imply that $z \leq G\left(y_k^l\right)$. The case $z < G\left(y_k^l\right)$ is handled as in the case $t_k^l > 0$ above. Suppose that $z = G\left(y_k^l\right)$. Then, for any $\gamma > 0$ such that $t_k^l + \gamma < t_k$, for sufficiently large n the mass expended in $\left(t_k^l, t_k^l + \gamma\right)$ by all players is smaller than $\varepsilon/6L$, because $A\left(t\right) = G\left(y_k^l\right)$ for any rational $t \in \left(t_k^l + \gamma, t_k\right)$. And if for sufficiently large n the players with types x for which $t_k^l \notin BR_x\left(\varepsilon\right)$ bid precisely t_k^l with probability at least $\varepsilon/6L$, then the ranking of a player who ties at t_k^l and loses is at most $G\left(y_k^l\right) - \varepsilon/12L$. But in this case each player of type x for which $t_k^l \notin BR_x\left(\varepsilon\right)$ would strictly prefer bidding slightly more than any bid in BR_x to bidding t_k^l, a contradiction.

To show (ii), note that if $t_k = t_{k'}^l$ for some k', then (i) follows from (ii). Thus, suppose that $t_k \neq t_{k'}^l$ for any k'. Suppose in contradiction that for any $\gamma > 0$ there are arbitrarily large n such that the mass expended in $\left(t_k - \gamma, t_k\right]$ by players with types x for which $t_k \notin BR_x\left(\varepsilon\right)$ is at least $\varepsilon/3L$. Take γ small enough so that the payoff that such players obtain by bidding slightly more than any bid in BR_x is higher by $\delta/2$ than the payoff that they would obtain by bidding $t_k - \gamma$ and getting y_k^u. Observe that for sufficiently large n, by bidding t_k any player almost certainly obtains a prize at most slightly better than $T^*\left(t_k\right) = y_k^u$. This because $t_k \neq t_{k'}^l$ for any k', so T^* is continuous at t_k and $T^*\left(t\right) > T\left(t_k\right)$ for $t > t_k$ (provided that $T\left(t_k\right) < 1$). Therefore, for large enough n a player with type x for which $t_k \notin BR_x\left(\varepsilon\right)$ would be better off bidding slightly above any $t \in BR_x$ than bidding in $\left(t_k - \gamma, t_k\right]$.

Part (iii) follows immediately from the fact that the value of A on $\left[t_k^l, t_k\right)$ is $G\left(y_k^l\right)$, by definition of t_k^l.

B.4 Proof of Lemma 9

As shown in the proof of Lemma 8, there exists a $\delta > 0$ such that for any type x any bid in the complement of $BR_x\left(\varepsilon\right)$ gives type x a payoff lower by at least δ than any bid in BR_x.

Take $\beta > 0$ such that for any type x, bid t, and prizes y' and y'' with $|y' - y''| \leq \beta$ we have

$$|U(x, y', t) - U(x, y'', t)| \leq \frac{\delta}{3}.$$

Next, take a K guaranteed by Lemma 7 for this β. In addition, take K large enough so the lengths of I_{K+1}, $I_{K+2}, ...$ sum up to less than $\beta/2$. Finally, for any $\lambda > 0$ take an N_λ that satisfies the definition of uniform convergence up to β on the complement of O_λ. (Note that K is the same for all λ.)

Suppose to the contrary of the statement of the lemma that there is a $\gamma > 0$ and a subsequence of contests such that a player i in the n-th contest has a best response t_n to the strategies of the other players that does not belong to $BR_{x_i^n}(\varepsilon) \cup \bigcup_{k=1}^{K} (t_k^l - \gamma, t_k)$. As usual, we assume that the subsequence is the entire sequence; moreover, we assume that $x_i^n \to x^*$ and $t_n \to t^*$.

Consider the following two cases:

A. ($t^* \neq t_k$ for any $k = 1, ..., K$) In this case, for some $\lambda > 0$ there is a neighborhood of t^* that is disjoint from O_λ. By uniform convergence of T^n to T^* up to β on the complement of O_λ,

$$U(x_i^n, T^n(t_n), t_n) - U(x_i^n, T^*(t_n), t_n) \leq \frac{\delta}{3}$$

for sufficiently large $n \geq N_\lambda$. And because $t_n \notin BR_{x_i^n}(\varepsilon)$, for any $t \in BR_{x_i^n}$ we have

$$U(x_i^n, T^*(t), t) - U(x_i^n, T^*(t_n), t_n) \geq \delta.$$

Thus, we obtain

$$U(x_i^n, T^*(t), t) - U(x_i^n, T^n(t_n), t_n) \geq \frac{2\delta}{3}.$$

Observe that any bid t' higher than t guarantees, for sufficiently large n, a prize not much worse than $T^*(t)$ with arbitrarily high probability.[31]

We will now show that by bidding t_n, for sufficiently high n type x_i^n obtains with arbitrarily high probability a prize no better than $T^n(t_n) + \beta$. Indeed, since t^* does not belong to

[31] To see why, note first that $t \notin (t_m^l, t_m)$ for any m. Next, consider the following two cases:

a) ($t = t_m$ for some m) Then: either (i) G has no atom at y_m^u and $A(t') > G(y_m^u)$ for any $t' > t$, or (ii) G has an atom at y_m^u and $A(t') \geq G(y_m^u)$ for any $t' > t$. In both these cases, any $t' > t$ guarantees (for sufficiently large n, and with high probability) a prize no lower than $T^*(t) = y_m^u$.

b) ($t \neq t_m$ for any m) In this case, $T^*(t) = \lim_{z \uparrow t} T(z)$, so by left-continuity of G^{-1}, an agent obtains a prize not much lower than $T^*(t)$ by bidding $t' = t$.

$\left[t_k^l, t_k\right]$ for any $k \leq K$, we have that $A\left(t'\right)$ is bounded away from $G\left(y_1^l\right), \ldots, G\left(y_K^l\right)$ for rationals t' sufficiently close to t^*. Therefore, $A^n\left(t_n\right)$ is also bounded away from $G\left(y_1^l\right), \ldots, G\left(y_K^l\right)$ for sufficiently large n. And for sufficiently large n, bidding t_n in the n-th contest gives with arbitrarily high probability a rank order arbitrarily close to $A^n\left(t_n\right)$. Since the lengths of I_{K+1}, I_{K+2}, ... sum up to less than $\beta/2$, for sufficiently large n, by bidding t_n a player obtains with arbitrarily high probability a prize no better than $G^{-1}\left(A^n\left(t_n\right)\right) + \beta$.

Therefore, by definition of β, we have that by bidding t_n type x_i^n obtains a payoff that is higher than $U\left(x_i^n, T^n\left(t_n\right), t_n\right)$ by at most slightly more than $\delta/3$. Consequently, for sufficiently large n player i would obtain by bidding some $t' > t$ in the n-th contest a payoff strictly higher than by bidding t_n, a contradiction.

B. ($t^* = t_k$ for some $k = 1, ..., K$) Then, consider a t^{**} slightly higher than t^*, such that t^{**} does not belong to $[t_k^l, t_k]$ for $k = 1, ..., K$, and such that: (i) for sufficiently large n the payoff (of any player) in the n-th contest of bidding t^{**} is lower than the payoff of bidding t_n by at most $\delta/6$; (ii) for sufficiently large n, we have that $U(x_i^n, T^*(t), t) - U(x_i^n, T^*(t^{**}), t^{**}) \geq 5\delta/6$ for any t in $BR_{x_i^n}$. This latter condition is possible because, by definition, $(x^*, t^*) \notin BR\left(\varepsilon\right)$, so for large enough n by continuity of U and the maximal payoff we have $U(x_i^n, T^*(t), t) - U(x_i^n, T^*(t^*), t^*) \geq 11\delta/12$ for any t in $BR_{x_i^n}$, and by right continuity of T^* at t^* we have that $U(x_i^n, T^*(t^*), t^*) - U(x_i^n, T^*(t^{**}), t^{**}) \leq \delta/12$ for t^{**} sufficiently close to t^*.

Now apply an argument analogous to that in case A with t^{**} playing the role of t_n to show that for large n a player of type x_i^n obtains at least $2\delta/3$ more by bidding t in $BR_{x_i^n}$ than by bidding t^{**}, and conclude that by (i) the same holds for t_n instead of t^{**} with $2\delta/3$ replaced with $\delta/2$, a contradiction.

B.5 Proof of Lemma 10

Monotonicity of the correspondence follows from strict single crossing, and upper hemicontinuity follows from standard arguments.[32]

Suppose that BR_x contains a pair of bids $t_1 < t_2$. Below we will show that for any $\varepsilon > 0$ and any interval $[a, b]$ such that $t_1 < a$ and $b < t_2$, for sufficiently large n the mass expended in $[a, b]$ by all players is at most ε. This implies that the function A, and therefore

[32] More precisely, this follows from the fact that BR_x is the set of all t such that $(t, T^*(t))$ maximizes type x's utility over the closure of the graph of T^*, which is a compact set.

T^*, is constant on every such interval $[a,b]$, and therefore on (t_1,t_2). But $T^*(t_2) > T^*(t_1)$ because $t_1 < t_2$ are in BR_x, so by definition of the discontinuity points t_k of T^* we must have $(t_1,t_2) \subseteq \left(t_k^l, t_k\right)$ for some k. And because $BR_x \subseteq B \setminus \cup \bigcup_{k=1}^{\infty} (t_k^l, t_k)$, we have that $t_1 = t_k^l$ and $t_2 = t_k$.

It remains to show that for any $\varepsilon > 0$, for sufficiently large n the mass expended in $[a,b]$ by all players is at most ε. We will show this for $\varepsilon/2$ and players of types lower than x (a similar argument applies to types higher than x). Choose $x' < x$ such that $F(x) - F(x') < \varepsilon/3$. For sufficiently small $\lambda > 0$, we have that $\sup \cup_{z \le x'} BR_z(\lambda) < a$ (because the correspondence is monotonic and $x' < x$, so every bid in $BR_{x'}$ is at most $t_1 < a$).

Therefore, by Lemma 9, there is some K such that for every $\gamma > 0$ and sufficiently large n any bid in $[a,b]$ made by a player of type $z \le x'$ in the n-th contest is in $\bigcup_{k=1}^{K} (t_k^l - \gamma, t_k)$. Consider one of these K intervals for which $(t_k^l - \gamma, t_k) \cap [a,b]$ is not empty. First note that $\sup \cup_{z \le x'} BR_z(\lambda) < a \le t_k$, so t_k is not in $\cup_{z \le x'} BR_z(\lambda)$. If $t_k^l > \sup \cup_{z \le x'} BR_z(\lambda)$, then by Lemma 8 there exists a γ such that for sufficiently large n the mass expended in $\left(t_k^l - \gamma, t_k\right)$ (and therefore $[a,b]$) by players of type $z \le x'$ is less than $\varepsilon/6K$. If $t_k^l \le \sup \cup_{z \le x'} BR_z(\lambda)$, then by (ii) and (iii) in Lemma 8, for sufficiently large n the mass expended in $[a, t_k]$ by players of type $z \le x'$ is less than $\varepsilon/6K$.

Therefore, for large enough n the mass expended in $[a,b]$ by players of type $z \le x'$ is smaller than $\varepsilon/6$, and because $F(x) - F(x') < \varepsilon/3$, the mass expended in $[a,b]$ by players of type $z \le x$ is smaller than $\varepsilon/2$.

B.6 Proof of Corollary 4

Choose $\varepsilon > 0$. Lemma 10 implies that there is a $\delta > 0$ and a finite number of intervals of types with total F-mass $\varepsilon/2$, such that for every type x not in one of these intervals, $BR_x(\delta) \subseteq [br(x) - \varepsilon, br(x) + \varepsilon]$.[33] Consider the fraction $1 - \varepsilon/2$ of players i for whom $BR_{x_i^n}(\delta) \subseteq [br(x_i^n) - \varepsilon, br(x_i^n) + \varepsilon]$, and let K be the one in the statement of 9 with δ instead of ε. By 9 with δ instead of ε and Lemma 8 with $\varepsilon^2/2$ instead of ε and $L = K$, at

[33] There is a $K > 0$ such that $\sum_{k>K} \left(t_k - t_k^l\right) < \varepsilon$. For each $k \le K$ such that $BR_{x_k} = \left\{t_k^l, t_k\right\}$ for some type x_k, consider the interval of types $[x_k - \lambda, x_k + \lambda] \cap [0,1]$, where λ is such that (the continuous) F increases by no more than $\varepsilon/2K$ on any interval no larger than 2λ. The sum of the F-mass of these intervals is no larger than $\varepsilon/2$, and the sum of the "jumps" of br on the complement of these intervals is no larger than ε.

most a fraction $\varepsilon/2$ of these players i bid outside of $[br(x_i^n)-\varepsilon, br(x_i^n)+\varepsilon]$ with probability higher than ε.

B.7 Proof of Lemma 11

Consider first some type $x > 0$ such that $x^{\min} = \inf\{z : br(z) = br(x)\} > 0$ and $x^{\max} = \max\{z : br(z) = br(x)\} < 1$ ($x^{\max}$ is well defined because br is left continuous).

If $br(x) \neq t_k$ for every k, then by Lemma 10 we have that $BR_{x^{\min}}$ is singleton, br is continuous at $x^{\min}$, and T^* is continuous at $br(x)$. Therefore, the arguments in the proof of Lemma 5, with Corollary 4 instead of Corollary 3, show that $T^*\left(br\left(x^{\min}\right)\right) = G^{-1}\left(F\left(x^{\min}\right)\right)$. If $br(x) = t_k$ for some k, then by Lemma 10 we have that $BR_{x^{\min}} = \left\{t_k^l, t_k\right\}$. Consider a player who bids some rational t in $\left(t_k^l, t_k\right)$. Corollary 4 implies that for sufficiently large n this player (1) outbids with arbitrarily high probability an arbitrarily large fraction of the players with types $x^{\min}$ or lower, so $\lim A(t) \geq F\left(x^{\min}\right)$, and (2) is outbid by an arbitrarily large fraction of the players with types higher than $x^{\min}$, so $A(t) \leq F\left(x^{\min}\right)$. Thus, $A(t) = F\left(x^{\min}\right)$, so we have $T^*\left(br\left(x^{\min}\right)\right) = T^*\left(t_k^l\right) = T^*(t) = G^{-1}(A(t)) = G^{-1}\left(F\left(x^{\min}\right)\right)$.

If $br(x) = t_k^l$ for some k, then $BR_{x^{\max}} = \left\{t_k^l, t_k\right\}$, and an argument similar to the one in the previous paragraph can be used to show that $T^*(br(x^{\max})) = G^{-1}(F(x^{\max}))$. Suppose that $br(x) \neq t_k^l$ for every k, so $BR_{x^{\max}}$ is a singleton and br is right continuous at $x^{\max}$. The arguments in the proof of Lemma 5, with Corollary 4 instead of Corollary 3, show that $G^{-1}(F(x^{\max})) \leq T^*(br(x^{\max})) \leq \lim_{\delta\downarrow 0} G^{-1}(F(x^{\max}+\delta))$. Suppose that $T^*(br(x^{\max})) \neq G^{-1}(F(x^{\max}))$. Then $F(x^{\max})$ is a point of discontinuity of G^{-1}, so $T^*(br(x^{\max})) = \lim_{\delta\downarrow 0} G^{-1}(F(x^{\max}+\delta)) = y_k^u$ for some k, and $br(x^{\max}) = t_k$. Thus, $G^{-1}(F(x^{\max})) \leq y_k^u$. Because $br(x) = br(x^{\max}) = t_k$, the previous paragraph shows that $A(t) = F\left(x^{\min}\right)$ for any t in $\left(t_k^l, t_k\right)$, so $G\left(y_k^l\right) = \lim_{t\uparrow t_k} A(t) = F\left(x^{\min}\right)$, where the first inequality follows from Lemma 6. Since $G\left(y_k^l\right) = F\left(x^{\min}\right)$ and $x^{\min} < x^{\max}$, we have $G^{-1}(F(x^{\max})) \geq y_k^u$. Together with the opposite inequality and $T^*(br(x^{\max})) = y_k^u$ shown above, we have that $T^*(br(x^{\max})) = G^{-1}(F(x^{\max}))$.

Thus, $T^*\left(br\left(x^{\min}\right)\right) = G^{-1}\left(F\left(x^{\min}\right)\right)$ and $T^*(br(x^{\max})) = G^{-1}(F(x^{\max}))$. Therefore, if $br(x) = br\left(x^{\min}\right)$, we have that $T^*(br(x)) = G^{-1}(F(x))$, as in the proof of Lemma 5 (recall that $br(x) = br(x^{\max})$ because br is left continuous). If $br(x) > br\left(x^{\min}\right)$, then arguments similar to the ones in the previous paragraph show that $T^*(br(x)) = G^{-1}(F(x))$.

The remaining cases can be handled similarly to the same cases in the proof of Lemma 5.

B.8 Proof of Lemma 12

In what follows we consider players with type x such that $F(x) \geq \varepsilon/4$. This guarantees that there is a strictly positive amount, independent of n, that all players are willing to pay in order to improve the ranking of the prize they get by $\delta/2$.

Take a K' large enough so that the lengths of $I_{K'+1}$, $I_{K'+2}, \ldots$ sum up to less than $\delta/2$. Take also a positive

$$\delta' < \min\{\delta, t_k - t_k^l : k = 1, \ldots, K'\}$$

such that

$$U(x, y', t') \geq U(x, y, t) \tag{10}$$

for any x, y, t with $y' - y > \delta/2$ and $t' - t \leq \delta'$.

In addition, take δ' small enough so that if $t \notin (t_k^l, t_k)$ for all $k = 1, \ldots, K'$ and $r \leq t + \delta'$, then $T^*(r) - T^*(t) \leq \delta/2$.

If strict single crossing holds, as in the proof of Corollary 4 take $\lambda > 0$ small enough such that there is a finite number of intervals of types with total F-mass $\varepsilon/4$, such that for every type x not in one of these intervals, $BR_x(\lambda) \subseteq (br(x) - \delta', br(x) + \delta')$. If strict single crossing does not hold, take $\lambda = \delta'$.

Finally, take K large enough so that for any $\gamma > 0$, if n is sufficiently large, the equilibrium bid of every player i in the n-th contest belongs with probability 1 to

$$BR_{x_i^n}(\lambda) \cup \bigcup_{k=1}^{K} (t_k^l - \gamma, t_k).$$

In addition, assume that $K \geq K'$.

Our first claim is that:

Given a $\gamma > 0$, for any $t \notin (t_k^l - \gamma, t_k)$ for all $k = 1, \ldots, K$, there exists an N_t such that for every $n \geq N_t$, a player for whom t is a best response and bids t obtains (with high probability) a prize y such that $|y - T^*(t)| < \delta/2$. We will also show that there exists an $N = N_t$ that is common for all such bids t.

Suppose first that $t \neq t_k$ for any $k = 1, \ldots, K$. Since $A(t) \neq G(y_k^l)$ and $A(t) \neq G(y_k^u)$ for any $k = 1, \ldots, K$, any rank order close to $A(t)$ also differs from $G(y_k^l)$ and $G(y_k^u)$. By (7), for sufficiently large n, a player who bids t has (with high probability) a rank order close to $A(t)$; in particular, this rank order differs from $G(y_k^l)$ and $G(y_k^u)$. By the assumption that the lengths of $I_{K+1}, I_{K+2}, ...$ sum up to less than $\delta/2$, this implies that the difference between $T^*(t)$ and the prize obtained by a player who bids t is at most $\delta/2$ (with high probability).

Suppose that $t = t_k$ for some $k = 1, ..., K$. By an argument analogous to the one used in the previous case, the prize obtained by a player who bids t cannot, as n increases, exceed $T^*(t)$ by more than $\delta/2$ with a probability that is bounded away from 0. And $T^*(t)$ cannot exceed this prize by more than $\delta/2$ with a probability that is bounded away from 0 as n increases, because the player would profitably deviate by bidding slightly above t, which would guarantee a prize no worse than $T^*(t)$ with arbitrarily high probability.

Now, note that the number N that was chosen for any bid t has the required property also for all bids close enough to t; in the case of $t = t_k$ for some $k = 1, ..., K$, we mean bids close enough and higher than t. That is, for every t there is a neighborhood W_t of that t with N_t that is common for all bids from this neighborhood. The family of sets W_t is an open covering of the compact set of bids t that satisfy $t \notin (t_k^l - \gamma, t_k)$ for $k = 1, ..., K$. Thus, it contains a finite subcovering, and any number N that exceeds numbers N_t for all elements of this finite subcovering has the required property.

Our first claim shows that part (1) of the lemma holds for bids $t \notin (t_k^l - \gamma, t_k)$ for all $k = 1, ..., K$. If strict single crossing holds, then part (2) of the lemma also holds for such bids. That is, the D^n-measure of (x, y, t) with $t \notin (t_k^l - \gamma, t_k)$ for all $k = 1, ..., K$ for which part (2) of the lemma is violated is $\varepsilon/2$. To see why, recall that λ was chosen so that for a fraction $1 - \varepsilon/2$ of the players i for large enough n we have that any t in $BR_{x_i^n}(\lambda)$ is in $(br(x_i^n) - \delta', br(x_i^n) + \delta')$, so $|t - r| < \delta' < \delta$ for $r = br(x_i^n)$. By our first claim, the prize y the player obtains by bidding t satisfies $|y - T^*(t)| < \delta/2$. Since r is in $BR_{x_i^n}$ and $|t - r| < \delta'$, by (10) we have that $T^*(t) - T^*(r) \leq \delta/2$. Finally, $T^*(r) - T^*(t) \leq \delta/2$, because of our choice of δ' and because $K \geq K'$.

Now consider bids t such that t is in $(t_k^l - \gamma, t_k)$ for some $k = 1, ..., K$. By (iii) of Lemma 8, we can disregard bids t in $(t_k^l + \gamma, t_k - \gamma)$ for some $k = 1, ..., K$. Suppose that t is in $(t_k - \gamma, t_k)$ for some $k = 1, ..., K$ such that $t_k \neq t_{k'}^l$ for all other $k' = 1, ..., K$. By (ii) of Lemma 8, it is with no loss of generality to assume that $t_k \in BR_{x_i^n}$.[34] We will show that for

[34] The lemma says only that the mass expended in $(t_k - \gamma, t_k]$ by types x for which $t_k \notin BR_x(\lambda)$ for some $\lambda > 0$ is small. However, if $\lambda > 0$ is sufficiently small, then the fraction of types x such that $t_k \notin BR_x$ but

sufficiently small γ and for sufficiently large n, player i obtains by bidding t (with arbitrarily high probability) a prize y in $(T^*(t_k)-\delta, T^*(t_k)+\delta)$. First, not that player i cannot obtain by bidding t a prize lower than $T^*(t_k)-\delta$ (with probability bounded away from 0 for arbitrarily large n), because for small enough γ it would be profitable to deviate to bidding slightly above t_k, and obtaining a prize not much lower than $T^*(t_k)$ with arbitrarily high probability. Player i cannot obtain by bidding t a prize higher than $T^*(t_k)+\delta$ (with probability bounded away from 0 for arbitrarily large n), because by (7), for any rational $r^m > t_k$ and sufficiently large n the rank order of player i is with arbitrarily high probability bounded above by $A(r^m)$. Thus, the upper bound on the prize follows from the assumption that $t_k \neq t^l_{k'}$ for all other $k' = 1, ..., K$, and that the lengths of I_{K+1}, I_{K+2}, ... sum up to less than $\delta/2$.

Finally, suppose that t is in $\left(t^l_k - \gamma, t^l_k + \gamma\right)$ for some $k = 1, ..., K$. By (i) of Lemma 8, it is with no loss of generality to assume that $t^l_k \in BR_{x^n_i}$. We will show that for sufficiently small γ and for sufficiently large n, equilibrium bidding in $\left(t^l_k - \gamma, t^l_k + \gamma\right)$ leads (with arbitrarily high probability) to a prize $y \in (T^*(t^l_k) - \delta, T^*(t^l_k) + \delta)$, except a small probability event. Indeed, by an argument similar to that from the previous case, such a bid cannot lead to a prize lower than $T^*(t^l_k) - \delta$ (with probability bounded away from 0 for arbitrarily large n). To obtain a prize higher than $T^*(t^l_k)+\delta$ with a non-vanishing probability, a player's expected rank order when bidding t cannot be, for large n, much lower than $G\left(y^l_k\right)$. But, as in the proof of Lemma 8, if non-vanishing fraction of players win a prize higher than $T^*(t^l_k)+\delta$ with a non-vanishing probability by bidding in $\left(t^l_k - \gamma, t^l_k + \gamma\right)$, then the increase in expected rank order on the interval $\left(t^l_k - \gamma, t^l_k + \gamma\right)$ is bounded away from 0 for all n, which contradicts the fact that $A^n(t^l_k + \gamma)$ approaches $G(y^l_k)$ as n increases.

C All-Pay Auctions with Head Starts

This setting is based on Siegel (2013a), who considered contests in which some players have an initial advantage relative to other players. In the single-agent setting, all non-zero prizes are identical and have the same value for all types. Type x has a head start of x, so he cannot bid less than x, and his cost of bidding $t \geq x$ is $t - x$. To model this and satisfy continuity, we set $U(x, y, t) = y - (t - x)$ and have the support of G be $\{0, 1\}$. We assume that F is uniform. We also assume that there is a mass 1/2 of non-zero prizes, so $G(y) = 1/2$ for all

$t_k \in BR_x(\lambda)$ is small.

$y \in [0,1)$ and $G(1) = 1$. We make these simplifying assumptions for ease of exposition.

All allocations are efficient. Not all of them, however, can be implemented by an IC, IR mechanism; one example is the allocation that assigns prizes to the types with the lowest head starts. The assortative allocation can be implemented, and prescribes for every type $x \leq 1/2$ prize 0 and for every type $x > 1/2$ prize 1. Any IC-IR mechanism that implements this allocation prescribes for every type $x \leq 1/2$ bid x and for every types $x \geq 1/2$ bid $3/2$.

We now show that the outcome of this mechanism approximates the equilibria of contests with many players and prizes. For every n, the n-th contest is an all-pay auction with $\ulcorner n/2 \urcorner$ (non-zero) prizes of value 1, in which the cost of bidding $x \geq i/n$ for player i is $x - i/n$. To simplify a little further, we consider $n = 2k+1$, so that there are $k+1$ identical non-zero prizes. As in some previous examples, the simplicity of the allocation and the IC-IR mechanisms that achieve it contrast with the relative complexity of players' equilibrium mixed strategies, which include atoms and gaps (see Siegel (2013a)). Moreover, while the IC-IR mechanism is given explicitly, players' mixed strategies are derived by an algorithm and are not described in closed form.

To show the approximation we make use of some equilibrium properties demonstrated by Siegel (2013a), and do not require a complete equilibrium characterization. We first observe that in equilibrium:

(a) players $i = 1, 2, ..., k-1$ (the players with the lowest head starts) bid their head starts and lose the contest (see Corollary 1 in Siegel (2013a)).

Now consider the other players. We will show that

(b) for any $\varepsilon > 0$, if k is sufficiently large, then the fraction of players $k, k+1, \ldots, 2k+1$ who bid $t \in [(3k+1)/(2k+1) - \varepsilon, (3k+1)/(2k+1)]$ with probability higher than $1 - \varepsilon$ is higher than $1 - \varepsilon$.

Indeed, note first that because players $1, 2, \ldots, k-1$ do not bid more than $(k-1)\,/\,(2k+1)$ and there are $k+1$ prizes, a player $i = k, k+1, \ldots, 2k+1$ wins a prize if she bids more than $(k-1)\,/\,(2k+1)$ and in addition outbids at least one of the other players $k, k+1, \ldots, 2k+1$. This implies that no player $i = k+1, \ldots, 2k+1$ bids more than $(3k+1)\,/\,(2k+1)$, because such bids are strictly dominated by 0 for player k. Another observation is that the equilibrium payoff of player $2k+1$ is $(k+1)\,/\,(2k+1)$, which is the difference between her head start and that of player k (see Theorem 1 or the first paragraph on page 16 of Siegel (2013a)).

Now suppose that for some $\varepsilon > 0$, for any $k > 0$ a fraction ε of the $k+1$ players $k, k+1, \ldots, 2k+1$ bid $t \in [(3k+1)/(2k+1) - \varepsilon, (3k+1)/(2k+1)]$ with probability at most $1-\varepsilon$. These players therefore bid less than $(3k+1)/(2k+1) - \varepsilon$ with probability at least ε (because players $k, k+1, \ldots, 2k+1$ do not bid more than $(3k+1) / (2k+1)$). But then by bidding $(3k+1)/(2k+1) - \varepsilon$ player $2k+1$ wins with a probability no lower than $1 - (1-\varepsilon)^{\varepsilon(k+1)}$ (because she wins whenever not all these players bid more than $(3k+1)/(2k+1) - \varepsilon$), so she can obtain a payoff no lower than

$$1 - (1-\varepsilon)^{\varepsilon(k+1)} - [(3k+1)/(2k+1) - \varepsilon - (2k+1)/(2k+1)]$$

$$= (k+1)/(2k+1) - (1-\varepsilon)^{\varepsilon(k+1)} + \varepsilon,$$

which exceeds $(k+1) / (2k+1)$ for sufficiently large k.

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