Bargaining with Wealth

Preliminary Draft

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ABSTRACT

I present a model in which randomly matched pairs of people bargain over the division of output in each period. Output can be consumed or stored for later consumption. People are identical except possibly in wealth (i.e., the stored output). The one-period utility is linear except for the starvation disutility (i.e., the negative utility under no consumption). The starvation disutility weakens the bargaining position of a poor person and strengthens that of a rich person in an otherwise symmetric bargaining, providing the incentive to accumulate wealth. A policy of preventing wealth accumulation can make both the rich and the poor become better off.

1. Introduction

In this paper, I present a stylized model that focuses on the bargaining motives of accumulating wealth. In the main version of the model, a unit of output, call it apple, is indivisible. In any period, eating an apple gives a fixed amount of utility but not eating any apples gives a negative starvation utility. Each person can store at most one apple over night. Then, a person is either rich (with one stored apple) or poor (no stored apple) at the beginning of each period. During each period, two people are randomly matched and can produce three apples. They bargain over the division of produced apples. The equilibrium bargaining outcome is for each person to take one apple and to take the third apple probabilistically. If the two people are either both rich or both poor, the probability of taking the third apple is one half for each person. If one person is rich and the other is poor, the probability of the rich taking the third apple is higher than one half since the outside option of the poor carries the starvation utility while that of the rich is to eat the stored apple. Under suitable parameter values, each person stores an apple if there are any left after consuming one apple. Nobody starves since each person will earn at least one apple in each period. Therefore, storing an apple is purely due to the bargaining motives. Further, storing apples is inefficient in that preventing the storing of apples increases each person's utility.

That a poor person with the outside option of starvation has a weaker bargaining position in relation to a rich person can be understood in terms of the risk preference: A more risk-averse person has more to lose from the failure of reaching the agreement, so has a weaker bargaining position. Kihlstrom, Roth, and Schmeidler (1981) shows that the more risk-averse party receives a less share of surplus in Nash bargaining: If person A's utility over the share of the surplus is an increasing and concave function of person B's utility, person A receives a less share of surplus than person B in an otherwise symmetric Nash bargaining. In the model of this paper, the one-period utility function over consumption is assumed to be identical across the population, but the presence of the starvation utility leads to an endogenously greater risk-averse preference over the income for a poor person than for a richer person.

In Krusell, Mukoyama, and Sahin (2010) and Bils, Chang, and Kim (2011), each worker has the the one-period utility function over consumption that exhibits the constant relative risk aversion (i.e., $u(c) = (c^{1-\gamma} - 1)/(1-\gamma)$ with $\gamma < 1$), bargains over the wage with the risk-neutral firm in each period, and can accumulate wealth. A worker with more wealth becomes "locally" less risk-averse and thus improves his bargaining position and obtains a higher wage. Thus there is a bargaining motive of accumulating wealth. The authors' objective is, however, to study the aggregate properties of the models under the productivity and unemployment shocks, which leads to the precautionary motive of accumulating wealth: Given the risk-averse utility function, wealth provides a self-insurance against a negative shock. The bargaining motive of wealth accumulation appears to play a minor role in their quantitative exercise. In the model of this paper, there is no precautionary motive for wealth accumulation since the equilibrium bargaining outcome guarantees everyone an income of at least one apple regardless of the consumption/saving behavior and the marginal utility of consumption is assumed to be constant beyond one apple. Also, production is modeled in terms of a decentralized random matching among identical people except possibly in wealth. The intent is to portray the bargaining motive of wealth accumulation and its welfare property abstracting from institutional arrangements that may cloud its logic.

2. The Model Economy

There are a large number of people, whose mass is normalized to one. Each person is randomly matched with another person every day. Each pair of matched people can produce three apples. The two partners of a match bargain over the division of produced apples. An apple is indivisible. Each person's one-period utility is $E\{U(c)\}$ where $U(c) = c \cdot I(c \geq 1) - \bar{u} \cdot I(c = 0)$. The parameter $\bar{u}$ has a positive value and means the starvation disutility. Each person has the access to a storage technology with the storage capacity of one apple. Each stored apple depreciates fully with probability δ overnight. The one-period budget constraint is $c + s = w + y$ where w is equal to one if an apple was stored yesterday and survived overnight and s is equal to one if an apple is stored today. Each person discounts the future utility by the rate β.

The outside option of each person in bargaining is not to produce, possibly consume a stored apple, and get rematched the next day. The equilibrium bargaining outcome features each partner taking at least one apple, so the crucial dimension of bargaining is each partner's probability of taking the third apple. If both partners are either rich (i.e., each has a stored apple) or poor (i.e., neither has a stored apple), the probability of taking the third apple is one half for each partner. If one partner is rich and the other is poor, the probability of taking the third apple is higher for the richer partner.

Since each person, regardless of his wealth level, is guaranteed at least one apple in each period in equilibrium, a person can avoid starvation regardless of his saving decision. Then, the marginal utility of consuming one more apple is constant and equal to one, so there is no precautionary motive to store an apple. Further, since each person discounts the future utility, it is optimal to consume all apples and not to store any if the division of produced apples is independent of whether a partner has a stored apple. In other words, the only reason for storing an apple is to raise the probability of taking the third apple in the next period.

2.1 The Benchmark: Equal Sharing of Output

The utility of a person with wealth w at the beginning of a period is:

$$V(w) = \sum_{w'} \left\{ \lambda(w') \sum_{y} \pi(y; w, w') \max_{s \in \Upsilon(w+y)} \left\{ u(w + y - s) + \tilde{V}(s) \right\} \right\} \tag{1}$$

where $w, w' \in \{0, 1\}$; $\lambda(w')$ is the share of population with w' apples; $\pi(y; w, w')$ is the probability of taking y apples from production given the person's wealth w and the partner's wealth w'; $\Upsilon(w + y) \equiv \{0, \min\{1, w + y\}\}$; and $\tilde{V}(s)$ is the utility of the person with wealth s at the end of the period:

$$\tilde{V}(0) = \beta V(0); \tag{2}$$

$$\tilde{V}(1) = \beta\delta \cdot V(0) + \beta(1 - \delta) \cdot V(1). \tag{3}$$

Suppose that each partner takes one apple for sure and takes the third apple with probability one half regardless of the partners' wealth levels: $\pi(1; w, w') = \pi(2; w, w') = 1/2$ for all w and w'. Since today's saving decision has no effect on the future income stream, the individual consumption/saving decision problem simplifies to maximizing utility given the exogenous stochastic income process. Note that the income will be at least one apple at all times; the utility is linear beyond one apple of consumption; and $\beta(1 - \delta) < 1$. Then, there is no reason to save any apples.

Proposition 1: Under the equal sharing of output (i.e., $\pi(1; w, w') = \pi(2; w, w') = 1/2$ for all w and w'), everyone consumes all of the apples, earned during the day or saved the day before, and saves none.

Proof: See the Appendix.

The proposition shows that any saving in this model is due to the effect of the wealth on the sharing of output. Further, since $\beta(1 - \delta) < 1$, any saving is inefficient in that the aggregate utility (i.e., the sum of discounted utility across all people) is maximized when each person consumes at least one apple and all apples are consumed in each period. The welfare impact of implementing a policy of no saving will be detailed in Section 2.3.

2.2 The Equilibrium: Unequal Sharing of Output

The two partners of a match bargain over the division of the produced apples. I assume the bargaining outcome to maximize the Nash product: for each (w, w'), $\{\pi(y; w, w')\}_y$ solves

$$\max\left\{\Big(\sum_y \pi(y;w,w')\cdot W(w+y)-W(w)\Big)\Big(\sum_y \pi(y;w,w')\cdot W(w'+3-y)-W(w')\Big)\right\} \quad (4)$$

where $W(w+y) \equiv \max_{s\in\Upsilon(w+y)}\{u(w+y-s)+\tilde{V}(s)\}$ is the post-bargain utility given the initial wealth w and the current-period income y. An equilibrium is the wealth distribution function $\lambda(w)$, the value functions, $V(w)$, $\tilde{V}(w)$, and $W(w+y)$, and the income-probability function, $\pi(y; w, w')$, that together satisfy (1) to (4). Below I consider three ranges of $\tilde{V}(1) - \tilde{V}(0)$ and derive the conditions on the parameter values that lead to each range.

Consume-All Equilibrium: $\tilde{V}(1) - \tilde{V}(0) < 1$

If $\tilde{V}(1) - \tilde{V}(0) < 1$, there will be no saving: $s = 0$ for all w and y. Further, $\lambda(0) = 1$. We have $\pi(1; w, w) = \pi(2; w, w) = 1/2$ for all w; and $W(w + y) = w + y + \tilde{V}(0)$ for all $w + y$. For the case of $(w, w') = (1, 0)$, I can take the first-order condition in (4) and show that there are two possible solutions to the maximization problem. One solution is for each partner to take one apple each and for one of the two partners to take the third apple probabilistically: $\pi(1; 1, 0) + \pi(2; 1, 0) = 1$ with the probability given by

$$\tilde{\pi} \equiv \pi(2; 1, 0) = \min\left\{1, \frac{1+\bar{u}}{2}\right\}. \quad (5)$$

Intuitively, the poor partner (i.e., $w = 0$) has a disadvantage in bargaining since his outside option carries the starvation utility while the outside option of the rich partner (i.e., $w = 1$) does not. As the disutility of starvation rises, the probability of the rich partner taking the third apple rises. Noting that $\pi(1; 0, 0) = \pi(2; 0, 0) = 1/2$ and $\lambda(0) = 1$, we have:

$V(0) = (1/2) \cdot (1 + \tilde{V}(0)) + (1/2) \cdot (2 + \tilde{V}(0))$; and $V(1) = (1 - \tilde{\pi}) \cdot (2 + \tilde{V}(0)) + \tilde{\pi} \cdot (3 + \tilde{V}(0))$. Then,

$$\tilde{V}(1) - \tilde{V}(0) = \beta(1-\delta)(V(1) - V(0)) = \beta(1-\delta)\left(\frac{1}{2} + \tilde{\pi}\right) < 1$$

iff

$$\tilde{\pi} < \frac{1}{\beta(1-\delta)} - \frac{1}{2}. \tag{6}$$

From (5) and (6), we have:

$$\min\left\{1, \frac{1+\bar{u}}{2}\right\} < \frac{1}{\beta(1-\delta)} - \frac{1}{2}. \tag{7}$$

The other solution to the maximization problem in (4) is for the poor partner to take one apple and for one of the two partners to take remaining two apples probabilistically: $\pi(0; 1, 0) + \pi(2; 1, 0) = 1$. This changes the expressions on the righthand sides of (5) and (6), but we obtain the same parameter restriction (7). Intuitively, once the poor partner consumes one apple, the marginal utility of consuming a remaining apple is constant, so the two variations of the lottery deliver the same expected utility to each partner and do not affect the equilibrium conditions.

There are no solutions to (4) other than the above two. In particular, it is not a solution for the rich partner (i.e., $w = 1$) to take two apples and for one of the two partners to take the remaining one apple probabilistically. Given the starvation utility, such an allocation can always be improved on (in terms of maximizing the Nash product) by guaranteeing one apple to the poor partner. In summary, we have:

Lemma 1: The consume-all equilibrium (i.e., $\tilde{V}(1) - \tilde{V}(0) < 1$), in which everyone is poor and consumes all of the apples earned during the day, exists if (7) holds. In the consume-all equilibrium, each partner takes at least one apple in any match. In the match of a poor person (i.e., a person with no stored apple) and a hypothetical rich person (i.e., a person who has deviated from the equilibrium by storing an apple), the probability of the rich person taking the third apple is higher than one half.

Proof: See the Appendix.

In (5), (6), and (7), Observe that the starvation utility $\bar{u}$ matters as a condition for the consume-all equilibrium by affecting the bargaining outcome of taking the third apple. As $\bar{u}$ rises, the probability of the rich partner taking the third apple rises, which raises the incentive to store an apple. On the other hand, a lower time discount rate β or a higher depreciation probability δ lower the incentive to store an apple. Thus the consume-all equilibrium requires the combination of a low enough starvation utility $\bar{u}$, a low enough time discount rate β, and a high enough depreciation probability δ.

Consume-Then-Save Equilibrium: $1 < \tilde{V}(1) - \tilde{V}(0) < 1 + \bar{u}$

If $1 < \tilde{V}(1) - \tilde{V}(0) < 1 + \bar{u}$, each person will save one apple if there any left after consuming one apple: $s = 0$ if $w+y \leq 1$; and $s = 1$ if $w+y \geq 2$. We have $\pi(1; w, w) = \pi(2; w, w) = 1/2$ for all w; $W(0) = -\bar{u} + \tilde{V}(0)$; $W(1) = 1 + \tilde{V}(0)$; and $W(w+y) = w + y - 1 + \tilde{V}(1)$ for all $w + y \geq 2$. Taking the first-order condition in (4), I can show that $\{\pi(y : 1, 0)\}_y$ is summarized as: $\pi(1; 1, 0) + \pi(2; 1, 0) = 1$ and

$$\tilde{\pi} \equiv \pi(2; 1, 0) = \min\left\{1, \frac{1}{2} \cdot \left(1 - \tilde{V}(1) + \tilde{V}(0) + \frac{1 + \bar{u}}{\tilde{V}(1) - \tilde{V}(0)}\right)\right\}. \tag{8}$$

We have:

$$V(0) = \left(\frac{1 - \tilde{\lambda}}{2} + \tilde{\lambda} \cdot \tilde{\pi}\right)(1 + \tilde{V}(0)) + \left(\frac{1 - \tilde{\lambda}}{2} + \tilde{\lambda} \cdot (1 - \tilde{\pi})\right)(1 + \tilde{V}(1)); \tag{9}$$

$$V(1) = \left((1 - \tilde{\lambda}) \cdot (1 - \tilde{\pi}) + \frac{\tilde{\lambda}}{2}\right)(1 + \tilde{V}(1)) + \left((1 - \tilde{\lambda}) \cdot \tilde{\pi} + \frac{\tilde{\lambda}}{2}\right)(2 + \tilde{V}(1)), \tag{10}$$

where $\tilde{\lambda} \equiv \lambda(1)$. The expressions inside large brackets are the transition probabilities, determined by the probability of meeting the rich (or the poor) and the probability of taking the third apple conditional on meeting the rich (or the poor). Since the equilibrium is assumed to be a steady state, tomorrow's rich people are today's rich people each of whom saves an apple, plus the subset of today's poor people each of whom saves an apple, discounted by the depreciation probability:

$$\tilde{\lambda} \equiv \lambda(1) = (1-\delta)\left(\tilde{\lambda} + (1-\tilde{\lambda})\left(\frac{1-\tilde{\lambda}}{2} + \tilde{\lambda}(1-\tilde{\pi})\right)\right). \tag{11}$$

Given the parameters, β, δ, and $\bar{u}$, we can use equations (2), (3), (8), (9), (10), and (11) to derive the equilibrium values of $\tilde{\pi}$, $\tilde{\lambda}$, and the value functions.

In order to find the conditions under which the consume-and-then save equilibrium exists, first consider the case of no depreciation: $\delta = 0$. In (11), observe that with no depreciation of stored apples, a rich person will never become poor so that there will only be the rich people in equilibrium: $\tilde{\lambda} = 1$ regardless of $\tilde{\pi}$. Then, from (2), (3), (9), and (10), we can derive:

$$\tilde{V}(1) - \tilde{V}(0) = \frac{1}{2} \cdot \frac{\beta}{1-\beta\tilde{\pi}}. \tag{12}$$

In (8) and (12), observe that as the starvation disutility $\bar{u}$ rises, both the probability of the rich person's taking the third apple $\tilde{\pi}$ and the benefit of wealth, $\tilde{V}(1) - \tilde{V}(0)$, rise. In order to have $\tilde{V}(1) - \tilde{V}(0) > 1$, in (12) we need the probability $\tilde{\pi}$ to be sufficiently high:

$$\tilde{\pi} > \frac{1}{\beta} - \frac{1}{2}. \tag{13}$$

Now turn to the case of a positive probability of depreciation: $\delta > 0$. With the possibility of losing the stored apple over night, there will be some poor people in equilibrium: $\tilde{\lambda} < 0$. Then, (11) can be rewritten as:

$$\tilde{\pi} = \frac{1}{2} + \frac{1}{2\tilde{\lambda}} - \frac{\delta}{1-\delta} \cdot \frac{1}{1-\tilde{\lambda}}. \tag{14}$$

Equation (14) implies a negative association between $\tilde{\pi}$ and $\tilde{\lambda}$. From (2), (3), (9), (10), and (14), I can derive:

$$\tilde{V}(1) - \tilde{V}(0) = \frac{\beta((1-\delta)/(2\tilde{\lambda}) - \delta)}{1-\beta+\beta\delta/(1-\tilde{\lambda})}. \tag{15}$$

Equation (15) implies a negative association between $\tilde{\lambda}$ and $\tilde{V}(1) - \tilde{V}(0)$. Thus (14) and (15) can be interpreted as follows. If the probability of the rich partner taking the third apple, $\tilde{\pi}$, were to rise exogenously, the chance of a poor person to become rich declines, which leads to a reduced fraction of the rich people in the population, $\tilde{\lambda}$, and a larger benefit of wealth, $\tilde{V}(1) - \tilde{V}(0)$. In (15), observe that there is $\lambda_0 \in (0, 1)$ such that $\tilde{V}(1) - \tilde{V}(0) > 1$ iff $\lambda < \lambda_0$. Solving for λ_0 in (15) and then substituting the expression of λ_0 in (14), we have

$$\tilde{\pi} > \pi_0 \equiv \frac{1}{\beta(1-\delta)} - \frac{1}{2} \tag{16}$$

iff $\lambda < \lambda_0$. Observe that (16) nests (13) as a special case.

Now, we can derive the conditions on the parameters for the consume-then-save equilibrium. Since $\tilde{\pi} \leq 1$, in (16) a necessary condition for an equilibrium with $\tilde{V}(1) - \tilde{V}(0) > 1$ is:

$$\frac{1}{\beta(1-\delta)} < \frac{3}{2}. \tag{17}$$

In order to have $\tilde{\pi} > \pi_0$ and $\tilde{V}(1) - \tilde{V}(0) > 1$, in (8) we need $\pi_0 < (1 + \bar{u})/2$ or:

$$\bar{u} > u_0 \equiv 2\left(\frac{1}{\beta(1-\delta)} - 1\right). \tag{18}$$

In summary, we have:

Lemma 2: The consume-then-save equilibrium (i.e., $1 < \tilde{V}(1) - \tilde{V}(0) < 1 + \bar{u}$), in which each person stores one apple if there are any left after consuming one apple, exists if (17) and (18) hold. In the consume-then-save equilibrium, each partner takes at least one apple in any match. In the match of a poor person (i.e., a person with no stored apple) and a rich person (i.e., a person with a stored apple), the probability of the rich person taking the third apple is higher than one half.

Proof: See the Appendix.

Note that (17) and (18) are the opposite of (7). Thus the consume-then-save equilibrium requires the opposite of the condition for the consume-all equilibrium: the combination of a high enough starvation disutility $\bar{u}$, a high enough time discount rate β, and a low enough depreciation probability δ, all of which raise the incentive to store an apple.

Save-Then-Consume Equilibrium: $\tilde{V}(1) - \tilde{V}(0) > 1 + \bar{u}$

If $\tilde{V}(1) - \tilde{V}(0) > 1 + \bar{u}$, each person will save one apple if there any apples, and then consume what remains: $s = 0$ if $w + y = 1$; and $s = 1$ if $w + y \geq 1$. I can can show that this save-then-consume equilibrium does not exist.

Lemma 3: The save-then-consume equilibrium (i.e., $\tilde{V}(1) - \tilde{V}(0) > 1 + \bar{u}$), in which each person stores one apple if there are any apples, and then consume any remaining apples, does not exist.

Proof: See the Appendix.

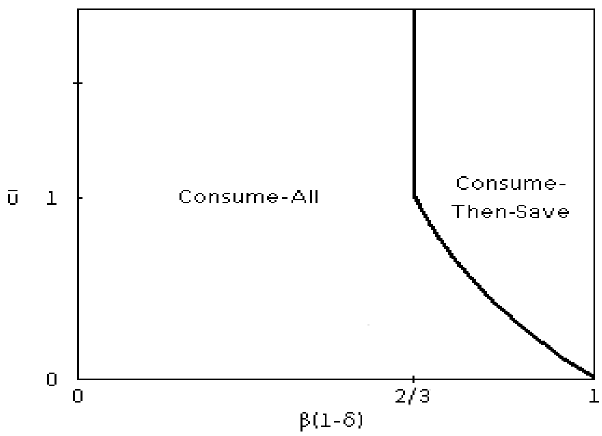
The intuition is that the incentive to store an apple in this economy is to avoid starvation even if there is no income tomorrow, thereby raising the bargaining position tomorrow. That $\tilde{V}(1) - \tilde{V}(0) > 1 + \bar{u}$ implies that a person stores an apple even at the cost of starving only to ensure that he continue storing that apple tomorrow. By induction, storing an apple is to ensure storing that apple forever. It is hard to imagine why such a behavior should strengthen the bargaining position of the person.

The above three cases of $\tilde{V}(1) - \tilde{V}(0)$ leaves two borderline cases, $\tilde{V}(1) - \tilde{V}(0) = 1$ and $\tilde{V}(1) - \tilde{V}(0) = 1 + \bar{u}$. First, consider the borderline case of $\tilde{V}(1) - \tilde{V}(0) = 1$: The incentive to store an apple is at the threshold so that each person is indifferent between consuming or storing an apple after having consuming one apple. Suppose that each person stores an apple with probability ϕ if there are any after consuming one. We can show that such an equilibrium, call it the consume-then-save-randomly, exists if (7) holds with equality. In the consume-then-save-randomly equilibrium, the probability of taking the third apple $\tilde{\pi}$ is the same as under the consume-all or the consume-then-save equilibrium (i.e., $\tilde{\pi}$ given by (5) or by setting $\tilde{V}(1) - \tilde{V}(0) = 1$ in (8)). Further, depending on ϕ, the fraction of rich people $\tilde{\lambda}$ can take on any value between the value under the consume-all equilibrium, equal to zero, and the value under the consume-then-save equilibrium, given by (11). Thus, the consume-then-save-randomly equilibrium can be viewed as a convex combination of the consume-all and the consume-then-save equilibria with $\phi = 0$ corresponding to the consume-all equilibrium and $\phi = 1$ to the consume-then-save equilibrium.

Now consider the other borderline case of $\tilde{V}(1) - \tilde{V}(0) = 1 + \bar{u}$: Each person is indifferent between consuming an apple to avoid starvation and storing it despite the starvation. Suppose that each person stores an apple with probability ϕ in such a situation. Using a variation of the proof of Lemma 3, we can show that such an equilibrium, call it randomly-save-then-consume equilibrium, does not exist under any parameter values.

The various cases of $\tilde{V}(1) - \tilde{V}(0)$ considered above show that the space of the parameters, β, δ, and $\bar{u}$, is divided into two zones, the zone of the consume-all equilibrium and the zone of the consume-then-save equilibrium with the threshold given by (7). Figure 1 below visualizes the two zones.

Figure 1: Equilibrium Zones



If (7) holds with equality, the incentive to store an apple is at the threshold (i.e., $\tilde{V}(1) - \tilde{V}(0) = 1$) and the consume-then-save-randomly equilibria, the convex combinations of the consume-all and the consume-then-save equilibria, exist.

Consider the comparative statics of raising the value of $\bar{u}$ starting from (the epsilon above) zero, holding β and δ. Initially, we are at the consume-all equilibrium zone. Both the probability of a rich person (who is on an off-equilibrium path since there are no rich people in equilibrium) taking the third apple, $\tilde{\pi}$, and the benefit of wealth, $\tilde{V}(1) - \tilde{V}(0)$, rise as $\bar{u}$ rises. If $\beta(1-\delta)$ is small enough so that (7) holds, $\tilde{\pi}$ reaches one eventually and a further rise in $\bar{u}$ does not affect the equilibrium. If $\beta(1-\delta)$ is large enough so that (7) holds, we would cross the threshold value $\bar{u}_0$ and switch to a consume-then-save equilibrium. Then, there will suddenly be some rich people in the economy ($\tilde{\lambda} = \lambda_0 > 0$). The probability of a rich person taking the third apple, $\tilde{\pi}$, is equal to $\tilde{\pi}_0$ when $\bar{u} = \bar{u}_0$, continues to rise as $\bar{u}$ rises, and reaches one at some value $\bar{u}_1$. Similarly, the benefit of wealth, $\tilde{V}(1) - \tilde{V}(0)$, continues to rise as $\bar{u}$ rises, and reaches some maximum value when $\bar{u} = \bar{u}_1$. Meantime, the fraction of the rich people $\tilde{\lambda}$ declines from λ_0 and reaches some minimum value $\tilde{\lambda}_1$ when $\bar{u}$ reaches $\bar{u}_1$. Thereafter, a further rise of $\bar{u}$ does not affect the equilibrium. In summay, we have:

Proposition 2: If (7) holds, the consume-all equilibrium with no storing of apples (i.e., $\tilde{V}(1) - \tilde{V}(0) < 1$) exists. If (7) holds with the opposite inequality, the consume-then-save equilibrium, in which each person stores one apple if there any left after consuming one apple (i.e., $1 < \tilde{V}(1) - \tilde{V}(0) < 1 + \bar{u}$), exists. If (7) holds with equality, the consume-then-save-randomly equilibrium, in which each person stores one apple randomly if there any left after consuming one apple (i.e., $\tilde{V}(1) - \tilde{V}(0) = 1$), exists. In any equilibria, each partner takes at least one apple in any match. In the match of a poor person (i.e., a person with no stored apple) and a rich person (i.e., a person with a stored apple), the probability of the rich person taking the third apple is higher than one half and rises to reach one as the starvation disutility $\bar{u}$ rises.

2.3 Welfare Comparison

Imagine the moment of making the consumption/saving decision (after the individual income has been realized) under the consume-then-save equilibrium, and consider introducing a (surprise) policy that prevents the storing of apples. The policy would force everyone to consume all apples and there would be an equal sharing of output from tomorrow as in the consume-all equilibrium.

In order to assess the welfare impact of the policy, first consider a would-be-rich person (i.e., a person who would have stored an apple under the consume-then-save equilibrium). From (2), (3), (9), and (10), the expected discounted utility of such a person under the consume-then-save equilibrium is:

$$\tilde{V}_s(1) = \frac{\beta}{1-\beta} \cdot \left(1 + (1-\tilde{\lambda}) \cdot \tilde{\pi} + \frac{\tilde{\lambda}}{2} - \frac{\delta}{\beta(1-\delta)} \cdot (\tilde{V}_s(1) - \tilde{V}_s(0))\right), \tag{19}$$

where the subscript s denotes the consume-then-save equilibrium. Under the policy of no saving, the expected discounted utility of the above would-be-rich person is the marginal utility of consuming a second or a third apple (instead of storing it), which is equal to one, plus the expected discounted utility under the equal sharing of output with no saving from tomorrow, which is equal to:

$$\tilde{V}_c(0) = \frac{3}{2} \cdot \frac{\beta}{1-\beta},$$

where the subscript c denotes the consume-all equilibrium. Let Ω denote the utility gain (loss) of the would-be-rich person upon the implementation of the no-saving policy:

$$\Omega \equiv 1 + \tilde{V}_c(0) - \tilde{V}_s(1). \tag{20}$$

If the depreciation probability $\delta = 0$, the fraction of rich people $\tilde{\lambda} = 1$ under the consume-then-save equilibrium so that $\tilde{V}_s(1) = 3/2 \cdot \beta/(1-\beta)$ in (19) and $\Omega = 1 > 0$ in (20). If $\delta > 0$, we can express $\tilde{V}_s(1)$ in terms of β, δ, and $\tilde{\lambda}$ only, using (14) and (15). Given β and δ, the remaining parameter $\bar{u}$ determines $\tilde{\lambda}$ in the range $[\lambda_1, \lambda_0]$, with a higher $\bar{u}$ leading to a lower $\tilde{\lambda}$, as discussed in Section 2.2. We can show that $\partial\tilde{V}_s(1)/\partial\tilde{\lambda} < 0$ within the range of $[\lambda_1, \lambda_0]$, which implies that $\tilde{V}_s(1)$ takes on the highest value when $\bar{u} \geq \bar{u}_1$ so that $\tilde{\lambda} = \lambda_1$. Let Ω_1 denote the value of Ω when $\bar{u} \geq \bar{u}_1$ given β and δ. Then, $\Omega \geq \Omega_1$. We can show that, somewhat surprisingly, $\Omega_1 > 0$ for any β and δ that satisfy (17). Then, $\Omega \geq \Omega_1 > 0$ for any β, δ, and $\bar{u}$ under which the consume-then-save equilibrium exists. Therefore, a would-be-rich person becomes better off upon the implementation of the no-saving policy.

Now, consider a would-be-poor person (i.e., a person who would not have stored an apple under the consume-then-save equilibrium). The expected discounted utility of such a person under the consume-then-save equilibrium is $\tilde{V}_s(0)$. The expected discounted utility of the person under the no-saving policy is $\tilde{V}_c(0)$. Since $\tilde{V}_s(1) - \tilde{V}_s(0) \geq 1$, we have:

$$\tilde{V}_s(0) \leq \tilde{V}_s(1) - 1 = \tilde{V}_c(1) - \Omega < \tilde{V}_c(1). \tag{21}$$

Therefore, a would-be-poor people also becomes better off under the no-saving policy. In summary, we have:

Proposition 3: Suppose that the economy is at a moment of making the consumption/saving decision under the consum-then-save equilibrium, and that a policy of no-saving is implemented unexpectedly. Then, everyone in the economy becomes better off.

Proof: See the Appendix.

Proposition 3 starkly shows the inefficiency of saving in this model. Recall that the only motive for saving is to bargain for a greater share of output. Since the total output is fixed, the advantage of wealth in bargaining is at the disadvantage of other people. Storing of an apple enables a person to receive the third apple with the probability of more than one half when matched with a poor partner; it also enables the person to avoid receiving the third apple with probability of less than one half when matched with a rich partner. Thus, storing of an apple has a negative externality on both the poor and the rich people. Consider the limiting case of no depreciation ($\delta = 0$) to focus on the externality on the rich people. With $\delta = 0$, there are only rich people ($\tilde{\lambda} = 1$) and everyone receives the third apple with the probability of one half in equilibrium. If everyone stops storing apples, each person can consume one extra apple immediately and continue receiving the third apple with the probability of one half from tomorrow. Therefore, everyone becomes better off under the no-saving policy. Proposition 3 shows that the force of the externality is strong enough so that everyone becomes better off whenever the economy is in a consume-then-save equilibrium.

3. Extensions

The inefficiency of wealth accumulation in the model relies on the starvation disutility when a poor person earns no income, which leads to a disadvantage of a poor person in bargaining with a rich person and thus to an incentive to accumulate wealth. I made a number of simplifying assumptions in order to have the logic of the story clear. It would seem that these simplifying assumptions are not qualitatively crucial for the result of the inefficiency of wealth accumulation. In this section, I explore a couple of alternative assumptions and derive the conditions under which the equilibrium with a wealth accumulation exists.

3.1 Divisibility of Output

Suppose that the apple is divisible so that dividing the output by a fraction of an apple and storing a fraction of an apple are possible. The one-period utility is $E\{U(c)\}$ where

$U(c) = c \cdot I(c \geq 1) + (c - (1-c) \cdot \bar{u}) \cdot I(c < 1)$, where c is assumed to be any non-negative value. This utility function is an expansion of the utility function in Section 2, interpolating the utility values when c is not an integer. The one-period budget constraint is $c + s = w + y$ where $s, w \in [0,1]$.

I will only focus on the limiting case of no depreciation of stored apples (i.e., $\delta = 0$). Otherwise, the model environment is the same as in Section 2. I will construct an equilibrium where everyone maintains one stored apple (i.e., $\tilde{\lambda} = 1$) as in the consume-then-save equilibrium with no depreciation. In order to insure that there is no incentive for a person to deviate from the equilibrium by saving less than one apple, I will characterize the off-equilibrium path (i.e., the bargaining outcome and the consumption/saving pattern of a person with less than one saved apple).

The utility of a person with wealth w at the beginning of a period is:

$$V(w) = \sum_y \pi(y; w, 1) \max_{s \in \Upsilon(w+y)} \big\{ u(w + y - s) + \beta V(s) \big\} \tag{1$'$}$$

where $w \in [0,1]$ and $\Upsilon(w+y) \equiv [0, \min\{1, w+y\}]$. The bargaining outcome maximizes the Nash product: for each w, $\{\pi(y; w, 1)\}_y$ solves

$$\max \left\{ \Big(\sum_y \pi(y; w, 1) \cdot W(w+y) - W(w) \Big) \Big(\sum_y \pi(y; w, 1) \cdot W(4-y) - W(1) \Big) \right\} \tag{4$'$}$$

where $W(w+y) \equiv \max_{s \in \Upsilon(w+y)} \{u(w+y-s) + \beta V(s)\}$. An equilibrium is the value functions, $V(w)$ and $W(w+y)$, and the income-probability function, $\pi(y; w, 1)$, that together satisfy $(1)'$ and $(4)'$.

The conjectured equilibrium properties are:

$$1 \leq \beta(V(1) - V(0)) \leq 1 + \bar{u} \tag{22}$$

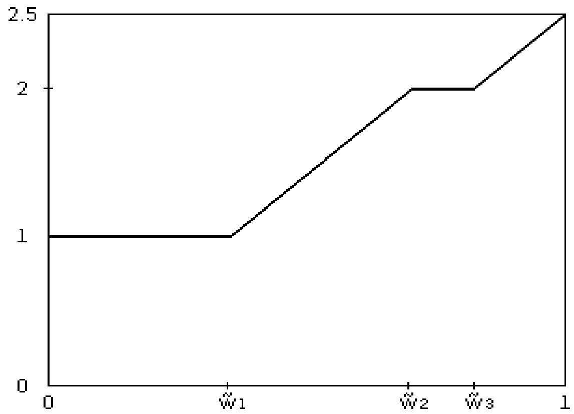
and

$$\beta(V(w) - V(0)) \leq w \cdot \beta(V(1) - V(0)) \tag{23}$$

for all $w \in [0, 1]$. Equation (22) implies an incentive to store an apple after consuming one. Equation (23) implies that everyone prefers a probabilistic storing of a whole apple to the certainty of storing a partial apple. Both of these properties mimic the consume-then-save equilibrium in Section 2.

Using the consumption/saving pattern given by (22) and(23), we can deduce the pattern of $W(w)$ and then, drawing the frontiers of output shares, deduce the pattern of the bargaining outcome. Figure 2 illustrates the expected value of the sum of wealth and income, $w + \sum_y y\pi(y; w, 1)$, as a function of wealth w.

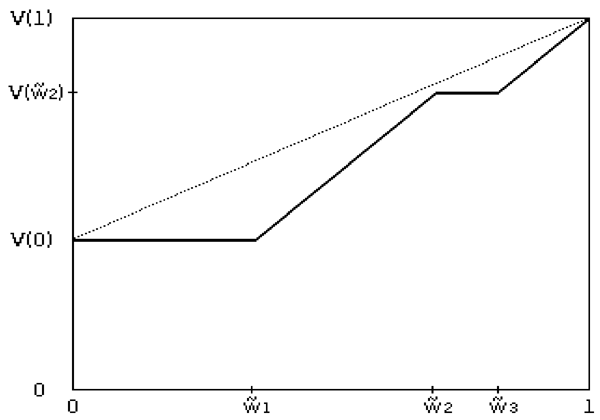
Figure 2: Expected Sum of Wealth and Income as a Function of Wealth



If a person's wealth is below $\tilde{w}_1$, his wealth has no value since his income will be just sufficient to avoid starvation regardless of his wealth. In maximizing $(4)'$, the solution is at a corner. If a person's wealth is between $\tilde{w}_1$ and $\tilde{w}_2$, his income is enough to consume an apple and to save an apple with some probability. Thus obtaining an apple beyond starvation is (endogenously) probabilistic reminiscent to the model in Section 2. If a person's wealth is between $\tilde{w}_2$ and $\tilde{w}_3$, his wealth is enough to consume an apple and to save an apple with certainty. In maximizing $(4)'$, the solution is at another corner. If a person's wealth is above $\tilde{w}_3$, his wealth is enough to consume more than an apple and to save an apple with certainty.

Using the pattern of the bargaining outcome, we can further deduce the pattern of the value function $V(w)$, illustrated in Figure 3.

Figure 3: Expected Utility as a Function of Wealth



In Figure 3, the dotted line is the righthand side of (23), multiplied by $1/\beta$. In order for (23) to be satisfied, the value function $V(w)$ must lie below the dotted line for all $w \in (0, 1)$. Observe that this is equivalent to $V(\tilde{w}_2)$ lying below the dotted line. Using this observation, we can show that (23) is equivalent to:

$$1 + \bar{u} \geq \frac{\beta}{2(1-\beta)}\left(\frac{\beta}{2(1-\beta)^2} - 1\right). \tag{24}$$

Further, given (24), (22) holds iff:

$$\beta \geq \frac{2}{3}. \tag{17$'$}$$

Equation (17)$'$ is equivalent to (17) for the case of $\delta = 0$ while (24) is more restrictive than (18). In summary, we have:

Proposition 4: An equilibrium with $\lambda(1) = 1$ exists if (17)$'$ and (24) hold.

Proof: See the Appendix.

Thus the equilibrium in which everyone maintains one stored apple (i.e., $\lambda(1) = 1$), as in the consume-then-save equilibrium with $\delta = 0$ in Section 2, exists if the valuation of future consumption is high enough (i.e., β large enough) and the starvation disutility is high enough (i.e., $\bar{u}$ large enough). The threshold disutility is higher than in Section 2.

We can conjecture possible equilibria when the starvation disutility $\bar{u}$ is below the threshold in (24). In Figure 3, the value function would lie above the dotted line. The bargaining outcome would then feature the probabilistic allocation of a fraction of an apple (instead of a whole apple) to a partner with additional flat segments in Figure 2, and the consumption/saving behavior would feature the storing of a fraction of an apple. Regardless of these complications in the off-equilibrium paths, the equilibrium with $\lambda(1) = 1$ may very well exist for a segment of $\bar{u}$ below the threshold in (24). In this sense, the characterization of the equilibrium in this section is for the special case of non-fractional saving, mimicking the equilibrium in Section 2.

3.2 Unlimited Storage Capacity

Now suppose that the storage capacity is unlimited so that the wealth w and the saving s can take on any non-negative integers $\{0, 1, 2, \ldots\}$. As in Section 3.1, assume that there is no depreciation of stored apples (i.e., $\delta = 0$). Otherwise, maintain the same environment as in Section 2. I will construct an equilibrium where everyone maintains a fixed number of stored apples, i.e., $\lambda(\bar{w}) = 1$ for some $\bar{w} \geq 1$. In order to insure that there is no incentive for a person to deviate from the equilibrium, I will characterize the off-equilibrium path. (i.e., the bargaining outcome and the consumption/saving pattern of a person with more or less than $\bar{w}$ stored apples).

The utility of a person with wealth w at the beginning of a period is:

$$V(w) = \sum_{y} \pi(y; w, \bar{w}) \max_{s \in \Upsilon(w+y)} \left\{ u(w + y - s) + \beta V(s) \right\} \qquad (1)''$$

where $w \in \{0, 1, \ldots\}$ and $\Upsilon(w+y) \equiv \{0, 1, \ldots, w+y\}$. The bargaining outcome maximizes the Nash product: for each w, $\{\pi(y; w, \bar{w})\}_y$ solves

$$\max \left\{ \Big(\sum_y \pi(y; w, \bar{w}) W(w+y) - W(w) \Big) \Big(\sum_y \pi(y; w, \bar{w}) \cdot W(\bar{w} + 3 - y) - W(\bar{w}) \Big) \right\} \quad (4)''$$

where $W(w + y) \equiv \max_{s \in \Upsilon(w+y)} \{u(w + y - s) + \tilde{V}(s)\}$. An equilibrium is the wealth level $\bar{w}$, the value functions, $V(w)$ and $W(w + y)$, and the income-probability function, $\pi(y; w, \bar{w})$, that together satisfy $(1)''$ and $(4)''$.

The conjectured equilibrium properties are: $\lambda(\bar{w}) = 1$ for some $\bar{w} \geq 1$;

$$1 \leq \beta(V(\bar{w} - n) - V(\bar{w} - n - 1)) \leq 1 + \bar{u} \tag{25}$$

for all $n \in \{0, 1, \ldots, \bar{w} - 1\}$;

$$\beta(V(\bar{w} - n) - V(\bar{w} - n - 1)) \leq \beta(V(\bar{w} - n - 1) - V(\bar{w} - n - 2)) \tag{26}$$

for all $n \in \{0, 1, \ldots, \bar{w} - 2\}$; and

$$\beta(V(\bar{w} + n) - V(\bar{w} + n - 1)) \leq 1 \tag{27}$$

for all $n \geq 1$. Equation (25) is a generalized version of the condition, $1 < \tilde{V}(1) - \tilde{V}(0) < 1 + \bar{u}$, for the consume-then-save equilibrium in Section 2, and implies an incentive to save an apple after consuming one when $w < \bar{w}$. Equation (26) ensures that the bargaining outcome takes the form of the probabilistic gain/loss of the third apple (rather than the probabilistic gain/loss of multiple apples) so that a person with $w < \bar{w}$ probabilistically accumulates one apple at a time. Equation (27) implies no incentive to store an apple beyond $\bar{w}$.

Given (25), (26), and (27), we can express $V(w)$ in terms of $\pi(y; w, \bar{w})$. Also, we can express $W(w)$ in terms of $V(w)$ and then, taking the first-order condition in $(4)''$, express

$\pi(y; w, \bar{w})$ in terms of $V(w)$. We can then show that the conditions $\beta(V(\bar{w})-V(\bar{w}-1)) > 1$ and $\beta(V(\bar{w}+1) - V(\bar{w})) < 1$ in (25) and (27) are equivalent to:

$$\frac{1}{\beta} - \frac{1}{2} \equiv \pi_0 \leq \breve{\pi}(\bar{w}-1) \leq \pi_1 \equiv \min\left\{1, \frac{1}{\beta} - \frac{1}{2} \cdot \frac{\beta}{2-\beta}\right\},$$

where $\breve{\pi}(\bar{w}-n) \equiv \pi(1; \bar{w}-n, \bar{w}) = \pi(2; \bar{w}, \bar{w}-n)$. In words, $\breve{\pi}(\bar{w}-n)$ is the probability of person with wealth $\bar{w}-n$ taking the third apple in a match with a person with the equilibrium wealth $\bar{w}$ or, equivalently, the probability of a person with $\bar{w}$ taking the third apple in a match with a person with $\bar{w}-n$. The first inequality is generalized version of (16) in Section 2: It is necessary for any equilibrium $\bar{w} \geq 1$. The second inequality was absent in Section 2 since the storage capacity was assumed in that section. Intuitively, a larger value of $\pi(2; \bar{w}, \bar{w}-1)$ implies a larger value of $\beta(V(\bar{w}) - V(\bar{w}-1))$. The outside option of a person with $\bar{w}$ in bargaining with a person with $\bar{w}+1$ is negatively affected by a higher value of $\beta(V(\bar{w}) - V(\bar{w}-1))$. Thus, a higher enough value of $\beta(V(\bar{w}) - V(\bar{w}-1))$ implies a small enough value of $\pi(2; \bar{w}, \bar{w}+1)$ and a large enough value of $\beta(V(\bar{w}+1) - V(\bar{w}))$. Since $\pi(2; \bar{w}, \bar{w}-1) \leq 1$, a necessary condition for the equilibrium with $\bar{w} \geq 1$ is:

$$\beta \geq \frac{2}{3}, \tag{17$'$}$$

which is an analog of (17) in Section 2.

In order to find the equilibrium values of $\bar{w}$, $\breve{\pi}(w)$, and $\bar{u}$, we can first derive:

$$\frac{\beta(1-\breve{\pi}(\bar{w}-n))}{1-\beta\breve{\pi}(\bar{w}-n-1)} = 2\breve{\pi}(\bar{w}-n) - 1 + \frac{\beta}{2(1-\beta\breve{\pi}(\bar{w}-1))}, \tag{28}$$

which yields $\breve{\pi}(\bar{w}-n-1)$ given $\breve{\pi}(\bar{w}-n)$. We can use (28) to derive the sequence of $(\breve{\pi}(\bar{w}-1), \breve{\pi}(\bar{w}-2), \ldots, \breve{\pi}(\bar{w}-\bar{n}))$ starting from any value of $\breve{\pi}(\bar{w}-1) \in [\pi_0, \pi_1]$. The value of $\breve{\pi}(\bar{w}-n)$ rises in n and there is a maximum value of n, call it $\bar{n}$, with $\breve{\pi}(\bar{w}-\bar{n}) \leq 1$. Then, for any $n \in \{1, 2, \ldots, \bar{n}\}$, we can set $\bar{w} = n$ so that $(\breve{\pi}(0), \breve{\pi}(1), \ldots, \breve{\pi}(\bar{w})) = (\breve{\pi}(\bar{w}-n), \breve{\pi}(\bar{w}-n+1), \ldots, \breve{\pi}(\bar{w}))$. Given, $\pi(0)$, $\bar{u}$ is given by:

$$\breve{\pi}(0) \leq \frac{1}{2}\left(1 - \beta(V(\bar{w}) - V(\bar{w}-1)) + \frac{1+\bar{u}}{\beta(V(1)-V(0))}\right), \tag{29}$$

with the strict equality if $\breve{\pi}(0) < 1$. Inequality (29) is a generalized version of (8). This algorithm of finding the equilibrium can be understood intuitively as follows. The bargaining advantage of a person with the equilibrium wealth is greater when matched with a person with a smaller wealth or, equivalently, a person with a smaller wealth has a greater disadvantage in bargaining with a person with the equilibrium wealth. These bargaining (dis)advantages provide the incentive to accumulate wealth. Since the probability of taking the third apple is bounded by one, the bargaining (dis)advantage is bounded and there is a maximum wealth sustainable in equilibrium.

Analyzing the above algorithm of finding the equilibrium, we can derive some other equilibrium properties. Observe that there are multiple equilibria along two dimensions, $\bar{w}$ and $\breve{\pi}(\bar{w}-1)$. For $\bar{w}=1$, any $\breve{\pi}(\bar{w}-1)=\breve{\pi}(0)\in[\pi_0,\pi_1]$ is an equilibrium. As $\bar{w}$ rises, the equilibrium values of $\breve{\pi}(\bar{w}-1)$ may be truncated from the above since the sequence of $\breve{\pi}(w)$ derived by (28) reaches the upper bound of one possibly sooner for a higher value of $\breve{\pi}(\bar{w}-1)\in[\pi_0,\pi_1]$. There is a maximum equilibrium wealth, call it $\bar{\omega}$, so that for any $\bar{w}>\bar{\omega}$, there is no range of $\breve{\pi}(\bar{w}-1)$ that supports $\bar{w}$ as an equilibrium. For any $\bar{w}\leq\bar{\omega}$, let $\pi_2(\bar{w})\in[\pi_0,\pi_1]$ denote the cutoff value so that $\bar{w}$ and any $\breve{\pi}(\bar{w}-1)\in[\pi_0,\pi_2(\bar{w})]$ are an equilibrium. Equation (28) maps the range $[\pi_0,\pi_2(\bar{w})]$ into a range of $\breve{\pi}(0)$, $[\breve{\pi}(0)|(\bar{w},\pi_0),\breve{\pi}(0)|(\bar{w},\pi_2(\bar{w}))]$, where the cutoff values, $\breve{\pi}(0)|(\bar{w},\pi_0)$ and $\breve{\pi}(0)|(\bar{w},\pi_2(\bar{w}))$, are higher for a higher $\bar{w}$ while they are smaller than one. Further, (29) maps the range $[\breve{\pi}(0)|(\bar{w},\pi_0),\breve{\pi}(0)|(\bar{w},\pi_2(\bar{w}))]$ into a finite range of $\bar{u}$, $[\check{u}(\bar{w}),\hat{u}(\bar{w})]$, if $\breve{\pi}(0)|(\bar{w},\pi_2(\bar{w})<1$, and into an infinite range of $\bar{u}$, $[\check{u}(\bar{w}),\infty)$, if $\breve{\pi}(0)|(\bar{w},\pi_2(\bar{w})=1$. The cutoff values, $\check{u}(\bar{w})$ and $\hat{u}(\bar{w})$, are higher for a higher $\bar{w}$, and if the range of $\bar{u}$ is infinite for some $\bar{w}$, it is infinite for a higher $\bar{w}$. In this sense, a greater starvation disutility $\bar{u}$ supports a higher equilibrium wealth level $\bar{w}$. Further, we can show that $\check{u}(\bar{w}+1)<\hat{u}(\bar{w})$ so that the ranges $\{[\check{u}(\bar{w}),\hat{u}(\bar{w})]\}_{\bar{w}}$, indexed by $\bar{w}$, overlap with each other. Thus, the ranges of $\bar{u}$ that support various values of $\bar{w}$ as equilibria form a single contiguous set. In summary, we have:

Proposition 5: If $(17)'$ holds, there is $\bar{\omega} \geq 1$ such that, for any $\bar{w} \in \{1, 2, \ldots, \bar{\omega}\}$, there is a range of $\bar{u}$, call it $\bar{U}(\bar{w})$, under which an equilibrium with $\lambda(\bar{w}) = 1$ exists. The range is either finite (i.e., $\bar{U}(\bar{w}) = [\check{u}(\bar{w}), \hat{u}(\bar{w})]$) or infinite (i.e., $\bar{U}(\bar{w}) = [\check{u}(\bar{w}), \infty)$). The range is increasing in $\bar{w}$ (i.e., $\check{u}(\bar{w}) < \check{u}(\bar{w}+1)$; $\hat{u}(\bar{w}) < \hat{u}(\bar{w}+1)$; and $\bar{U}(\bar{w}+1)$ is infinite if $\bar{U}(\bar{w})$ is infinite). Further, the ranges are contiguous (i.e., $\check{u}(\bar{w}+1) < \hat{u}(\bar{w})$) so that an equilibrium with some $\bar{w}$ exists if $\bar{U}(\bar{\omega})$ is finite and $\bar{u} \in [\check{u}(1), \hat{u}(\bar{\omega})]$, or if $\bar{U}(\bar{\omega})$ is infinite and $\bar{u} \in [\check{u}(1), \infty)$.

Proof: See the Appendix.

We can show that, for the case of $\bar{w} = 1$, the equilibrium range of $\bar{u}$ is:

$$\bar{U}(1) = \begin{cases} [\check{u}(1), \hat{u}(1)] = \left[2 \cdot \left(\dfrac{1}{\beta} - 1\right), \dfrac{8}{\beta} \cdot \left(\dfrac{1}{\beta} - 1\right)\right] & \text{if } \pi_1 < 1; \\ [\check{u}(1), \ \infty \ ) = \left[2 \cdot \left(\dfrac{1}{\beta} - 1\right), \qquad \infty \qquad \right) & \text{if } \pi_1 = 1. \end{cases}$$

The value of $\check{u}(1)$ is the same cutoff value as in (18) in Section 2. The difference is that, under the unlimited storage capacity assumed in this section, the equilibrium range of $\bar{u}$ may be truncated from the above in order to ensure that there is no incentive to save more than one apple.

References

Bills, M., Chang, Y., and Kim, S. (2011), “Worker Heterogeneity and Endogenous Separations in a Matching Model of Unemployment Fluctuations,” *American Economic Journal: Macroeconomics* 3:128-154.

Kihlstrom, R., Roth, A., and Schmeidler, D. (1981), “Risk Aversion and Solutions to Nash’s Bargaining Problem” in *Game Theory and Mathematical Economics*, edited by Moeschlin, O. and Pallaschke, D., Amsterdam: North-Holland.

Krusell, P., Mukoyama, T., and Sahin, A. (2010), “Labour-Market Matching with Precautionary Savings and Aggregate Fluctuations,” *Review of Economic Studies* 77:1477-1507.

Appendix: Proofs for Propositions and Lemmas

Proof of Proposition 1

We have

$$V(0) = \frac{1}{2} \max_{s\in\{0,1\}} \left\{U(1-s) + \tilde{V}(s)\right\} + \frac{1}{2} \max_{s\in\{0,1\}} \left\{U(2-s) + \tilde{V}(s)\right\};$$

$$V(1) = \frac{1}{2} \max_{s\in\{0,1\}} \left\{U(2-s) + \tilde{V}(s)\right\} + \frac{1}{2} \max_{s\in\{0,1\}} \left\{U(3-s) + \tilde{V}(s)\right\}.$$

Observe that

$$V(0) \geq \frac{1}{2} \cdot (U(1) + \tilde{V}(0)) + \frac{1}{2} \cdot (U(2) + \tilde{V}(0))$$

so that

$$\begin{aligned} V(1) - V(0) \leq\ & \frac{1}{2} \max_{s\in\{0,1\}} \left\{U(2-s) - U(1) + \tilde{V}(s) - \tilde{V}(0)\right\} \\ +\ & \frac{1}{2} \max_{s\in\{0,1\}} \left\{U(3-s) - U(2) + \tilde{V}(s) - \tilde{V}(0)\right\} \\ =\ & \max\{1, \tilde{V}(1) - \tilde{V}(0)\}. \end{aligned}$$

Then,

$$\tilde{V}(1) - \tilde{V}(0) = \beta(1-\delta)(V(1) - V(0)) \leq \beta(1-\delta) \max\{1, \tilde{V}(1) - \tilde{V}(0)\}$$

so that

$$\tilde{V}(1) - \tilde{V}(0) \leq \beta(1-\delta) < 1.$$

This implies that $s = 0$ and $c = w + y$ and for all w and y.

Proof of Lemma 1

Consider first-order conditions when $(w, w') = (1, 0)$ in (4). A first-order condition is $\pi(1; 1, 0) + \pi(2; 1, 0) = 1$ and

$$\begin{aligned} &(W(3) - W(2))(\tilde{\pi}W(1) + (1-\tilde{\pi})W(2) - W(0)) \\ &+(W(1) - W(2))(\tilde{\pi}W(3) + (1-\tilde{\pi})W(2) - W(1)) \geq 0 \end{aligned} \tag{A2}$$

with the strict equality if $\tilde{\pi} \equiv \pi(2;1,0) < 1$. Using that $W(w+y) = w + y + \tilde{V}(0)$ for all $w+y$, we can show that (A2) is equivalent to (5) in the main text. Derivation of (6) and (7) is as discussed in the main text.

Another first-order condition in (4) with $(w, w') = (1, 0)$ is: $\pi(0;1,0) + \pi(2;1,0) = 1$ and

$$\begin{aligned} &(W(3) - W(1))(\tilde{\pi}W(1) + (1-\tilde{\pi})W(2) - W(0)) \\ &+(W(1) - W(2))(\tilde{\pi}W(3) + (1-\tilde{\pi})W(1) - W(1)) \geq 0 \end{aligned} \tag{A3}$$

with the strict equality if $\tilde{\pi} \equiv \pi(2;1,0) < 1$. Repeating the above steps, we can show that (A3) is equivalent to

$$\tilde{\pi} \equiv \pi(2;1,0) = \min\left\{1, \frac{3+\bar{u}}{4}\right\} \tag{A4}$$

and $\tilde{V}(1) - \tilde{V}(0) < 1$ iff

$$\tilde{\pi} < \frac{1}{2\beta(1-\delta)} + \frac{1}{4}. \tag{A5}$$

From (A4) and (A5), we also have (7).

There are no first-order conditions, other than the above two, that yield additional solutions. In particular, consider the following first-order condition: $\pi(2;1,0) + \pi(3;1,0) = 1$ and

$$\begin{aligned} &(W(4) - W(3))(\bar{\pi}W(0) + (1-\bar{\pi})W(1) - W(0)) \\ &+(W(0) - W(1))(\bar{\pi}W(4) + (1-\bar{\pi})W(3) - W(1)) \leq 0 \end{aligned} \tag{A6}$$

with the strict equality if $\bar{\pi} \equiv \pi(3;1,0) > 0$. We can show that (A6) is equivalent to $\bar{\pi} = 0$. Thus, only the corner solution is possible, but this only completes the corner solution in (5) and (A4). Therefore, we have Lemma 1.

Proof of Lemma 2

Consider first-order conditions when $(w, w') = (1, 0)$ in (4). A first-order condition is $\pi(1;1,0) + \pi(2;1,0) = 1$ and (A2) with the strict equality if $\tilde{\pi} \equiv \pi(2;1,0) < 1$. Using that $W(0) = -\bar{u} + \tilde{V}(0)$; $W(1) = 1 + \tilde{V}(0)$; and $W(w+y) = w + y - 1 + \tilde{V}(1)$ for all $w + y \geq 2$, we can show that (A2) is equivalent to (8) in the main text. There are no first-order conditions, other than (8), that yield additional solutions. In particular, consider the following first-order condition: $\pi(2;1,0) + \pi(3;1,0) = 1$ and (A6) with the strict equality if $\bar{\pi} \equiv \pi(3;1,0) > 0$. We can show that (A6) is equivalent to $\bar{\pi} = 0$. Thus, only the corner solution is possible, but this only completes the corner solution in (8).

Derivation of (9), (10), and (11) is as discussed in the main text. From (2), (3), (9), and (10), we have:

$$\tilde{V}(1) - \tilde{V}(0) = \beta(1-\delta)(V(1) - V(0)) = \frac{\beta(1-\delta)(\tilde{\pi}(1-\tilde{\lambda}) + \tilde{\lambda}/2)}{1 - \beta(1-\delta)(\tilde{\pi}\tilde{\lambda} + (1-\tilde{\lambda})/2)}. \tag{A7}$$

If $\delta = 0$, $\tilde{\lambda} = 1$ in (11). Then, (A7) becomes (12). Derivation of (13) is as discussed in the main text. If $\delta > 0$, (11) and (A7) become (14) and (15), respectively.

In (8), $\tilde{\pi}$ is determined given $\tilde{V}(1) - \tilde{V}(0)$. If $\delta = 0$, $\tilde{V}(1) - \tilde{V}(0)$given $\tilde{\pi}$ in (12). If $\delta > 0$, $\tilde{\lambda}$ is determined given $\tilde{\pi}$ in (14), and $\tilde{V}(1) - \tilde{V}(0)$ is determined given $\tilde{\lambda}$ in (15). Thus (8), (12), (14), and (15) implicitly define a function, call it Γ, that takes a value of $\tilde{V}(1) - \tilde{V}(0)$ and returns another value of $\tilde{V}(1) - \tilde{V}(0)$. The equilibrium value of $\tilde{V}(1) - \tilde{V}(0)$ is the one that satisfies: $\Gamma(\tilde{V}(1) - \tilde{V}(0)) = \tilde{V}(1) - \tilde{V}(0)$.

Derivation of (16) and (17) is as discussed in the main text. Assume that (17) holds from now on. Consulting (8) and (16), let Δ be the value of $\tilde{V}(1) - \tilde{V}(0)$ that solves:

$$\pi_0 = \frac{1}{2} \cdot \left(1 - \Delta + \frac{1+\bar{u}}{\Delta}\right). \tag{A8}$$

Then, $\tilde{V}(1) - \tilde{V}(0) < \Delta$ in equilibrium if an equilibrium exists. A necessary condition for an equilibrium with $\tilde{V}(1) - \tilde{V}(0) > 1$ is $\Delta > 1$, which is equivalent to (18). Assume that (18) holds from now on. Let Λ be the value of $\tilde{V}(1) - \tilde{V}(0)$ that yields $\tilde{\pi} = 1$ in (8):

$$1 = \frac{1}{2} \cdot \left(1 - \Lambda + \frac{1+\bar{u}}{\Lambda}\right).$$

In (8), (12), (14), and (15), observe that $\Gamma(\tilde{V}(1) - \tilde{V}(0)) = \Gamma(\Lambda)$ for all $\tilde{V}(1) - \tilde{V}(0) \leq \Lambda$.

If $\Lambda < \tilde{V}(1) - \tilde{V}(0) < \Delta$, $\tilde{\pi}$ is decreases as $\tilde{V}(1) - \tilde{V}(0)$ increases in (8). If $\delta = 0$, $\tilde{V}(1) - \tilde{V}(0)$ decreases as $\tilde{\pi}$ decreases in (12). If $\delta > 0$, $\tilde{\lambda}$ increases as $\tilde{\pi}$ decreases in (14), and $\tilde{V}(1) - \tilde{V}(0)$ decreases as $\tilde{\lambda}$ increases in (15). Thus, Γ is decreasing in $\tilde{V}(1) - \tilde{V}(0)$ if $\Lambda < \tilde{V}(1) - \tilde{V}(0) < \Delta$. Consulting (16) and (18), we have $\Gamma(1) > 1$ and $\Gamma(\Delta) = 1$. Further, Γ is continuous and non-increasing if $1 < \tilde{V}(1) - \tilde{V}(0) < \Delta$. Therefore, there is a unique fixed point in $(1, \Delta)$. Noting that $\Delta < 1 + \bar{u}$ in (A8), we have Lemma 2.

Proof of Lemma 3

Consider first-order conditions when $(w, w') = (1, 0)$ in (4). A first-order condition is $\pi(1; 1, 0) + \pi(2; 1, 0) = 1$ and (A2) with the strict equality if $\tilde{\pi} \equiv \pi(2; 1, 0) < 1$. Using that $W(0) = -\bar{u} + \tilde{V}(0)$; $W(1) = -\bar{u} + \tilde{V}(1)$; and $W(w + y) = w + y - 1 + \tilde{V}(1)$, we can show that (A2) is equivalent to

$$\tilde{\pi} \leq \frac{1}{2} \cdot \left(\frac{\tilde{V}(1) + \tilde{V}(0)}{1 + \bar{u}} - \bar{u} \right). \tag{A9}$$

There are no first-order conditions, other than (A9), that yield additional solutions. In particular, consider the following first-order condition: $\pi(2; 1, 0) + \pi(3; 1, 0) = 1$ and (A6) with the strict equality if $\bar{\pi} \equiv \pi(3; 1, 0) > 0$. We can show that (A6) is equivalent to $\bar{\pi} = 0$. Thus, only the corner solution is possible, but this only completes the corner solution in (A9).

We have:

$$V(0) = \left(\frac{1 - \tilde{\lambda}}{2} + \tilde{\lambda} \cdot \tilde{\pi} \right)(-\bar{u} + \tilde{V}(1)) + \left(\frac{1 - \tilde{\lambda}}{2} + \tilde{\lambda} \cdot (1 - \tilde{\pi}) \right)(1 + \tilde{V}(1));$$

$$V(1) = \left((1 - \tilde{\lambda}) \cdot (1 - \tilde{\pi}) + \frac{\tilde{\lambda}}{2} \right)(1 + \tilde{V}(1)) + \left((1 - \tilde{\lambda}) \cdot \tilde{\pi} + \frac{\tilde{\lambda}}{2} \right)(2 + \tilde{V}(1)).$$

Since everyone saves one apple in equilibrium, $\tilde{\lambda} \equiv \lambda(1) = 1 - \delta$. Then, we have:

$$\tilde{V}(1) - \tilde{V}(0) = \beta(1-\delta)(V(1) - V(0)) = \beta(1-\delta)\left(\tilde{\pi}(1 + \bar{u}(1-\delta)) + \frac{1+\delta\bar{u}}{2}\right). \quad \text{(A10)}$$

Substituting (A10) in (A9), we have:

$$\tilde{\pi} \leq \frac{1}{2} \cdot \frac{\beta(1-\delta)(1+\delta\bar{u}) - 2\bar{u}(1+\bar{u})}{2(1+\bar{u}) - \beta(1-\delta)(1+\bar{u}(1-\delta))} < \frac{1}{2}. \quad \text{(A11)}$$

Substituting (A11) in (A10), we have:

$$\tilde{V}(1) - \tilde{V}(0) < \beta(1-\delta)\left(1 + \frac{\bar{u}}{2}\right) < 1 + \bar{u}.$$

This is a contradiction. Therefore, we have Lemma 3.

Proof of Proposition 3

It was shown in the main text that $\Omega > 0$ when $\delta = 0$. If $\delta > 0$, given (19), we can express $\tilde{V}_s(1)$ in terms of β, δ, and $\tilde{\lambda}$ only, using (14) and (15):

$$\tilde{V}_s(1) = \frac{\beta}{1-\beta} \cdot \left(1 + \left(\frac{1}{2\tilde{\lambda}} - \frac{\delta}{1-\delta}\right)\left(1 - \frac{1}{\beta/(1-\tilde{\lambda}) + (1-\beta)/\delta}\right)\right). \quad \text{(A12)}$$

Given β and δ, the remaining parameter $\bar{u}$ determines $\tilde{\lambda}$ in the range $[\lambda_1, \lambda_0]$, corresponding to a $\tilde{\pi}$ in the range $[\tilde{\pi}_0, 1]$ according to (14). A higher $\bar{u}$ leads to a higher $\tilde{\pi}$ and a lower $\tilde{\lambda}$ within the ranges. Taking the partial derivative of $\tilde{V}_s(1)$ with respect to $\tilde{\lambda}$, we can show:

$$\frac{\partial \tilde{V}_s(1)}{\partial \tilde{\lambda}} < \frac{\beta}{1-\beta} \cdot \frac{1}{2\tilde{\lambda}^2}\left(\frac{1}{\beta + (1-\beta)(1-\tilde{\lambda})/\delta} - 1\right) < 0$$

at any $\tilde{\lambda} \in [\lambda_1, \lambda_0]$. The second inequality uses the relation $1 - \tilde{\lambda} > \delta$ that can be derived from (14), (16), and (17) as follows. Define $\bar{\lambda}$ to be the solution to (14) assuming $\tilde{\pi} = 1/2$: $\bar{\lambda} = (1-\delta)/(1+\delta)$. Since $\tilde{\pi} > 1/2$ in (16), $\tilde{\lambda} < \bar{\lambda}$ in (14). Then, $1-\tilde{\lambda} > 1-\bar{\lambda} = 2\delta/(1+\delta) > \delta$ since $\delta < 1/3$ in (17). Thus, holding β and δ, $\tilde{V}_s(1)$ decreases in $\tilde{\lambda}$ or, equivalently, increases in $\bar{u}$. The maximum value of $\tilde{V}_s(1)$ is when $\tilde{\lambda} = \lambda_1$, $\tilde{\pi} = 1$, and $\bar{u} \geq \bar{u}_1$. Then, $\Omega \geq \Omega_1$, where Ω_1 denote the value of Ω when $\bar{u} \geq \bar{u}_1$ given β and δ. From (A12), we have:

$$\Omega_1 = 1 - \frac{\beta}{1-\beta} \cdot \left(\left(\frac{1}{2\lambda_1} - \frac{\delta}{1-\delta} \right) \left(1 - \frac{1}{\beta/(1-\lambda_1) + (1-\beta)/\delta} \right) - \frac{1}{2} \right). \tag{A13}$$

Setting $\tilde{\pi} = 1$ in (14), we can derive:

$$\delta = \frac{(1-\lambda_1)^2}{(1+\lambda_1^2)}. \tag{A14}$$

Note that λ_1 depends only on δ and not on β or $\bar{u}$. Now, we can substitute the above expression of δ in terms of λ_1 in the expression of Ω_1 in (A13), and take the partial derivative of Ω_1 with respect to β. Doing so, we can show that, for any δ with λ_1 given by (A14), Ω_1 takes the minimum value when

$$\beta = \max \left\{ \frac{2}{3(1-\delta)}, \frac{1}{1+\delta} - \left(\frac{1}{(1+\delta)^2} - \frac{1-\delta-\delta/(1-\lambda_1)}{(1+\delta)(1-\delta/(1-\lambda_1))} \right)^{1/2} \right\}. \tag{A15}$$

The first argument is the minimum given by (17) and the second argument is always less than 1. Finally, we can show that Ω_1 increases in δ in (A13) when λ_1 and β are given by (A14) and (A15), and $\Omega_1 \to 0$ as $\delta \to 0$. Therefore, $\Omega_1 > 0$ for any β and δ. Then, we have: $\Omega \geq \Omega_1 > 0$ for any β, δ, and $\bar{u}$. Therefore, a would-be-rich person becomes better off upon the implementation of the no-saving policy.

It is shown in the main text that a would-be-poor person also becomes better off, summarized in (21). Therefore, we have Proposition 3.

Proof of Proposition 4

Equations (22) and (23) are consistent with the following consumption/saving pattern: $c = w+y$ and $s = 0$ if $w+y \leq 1$; and $c = w+y-1$ and $s = 1$ if $w+y \geq 2$. As a proof, observe that $U(w+y)-U(w+y-s) = s(1+\bar{u}) \geq s \cdot \beta(V(1)-V(0)) \geq \beta(V(s)-V(0))$ if $s \leq w+y \leq 1$;

and $U(w+y-s) - U(w+y-1) = 1-s \leq (1-s)\cdot\beta(V(1)-V(0)) \leq \beta(V(1)-V(s))$ if $w+y \geq 2$ and $s \leq 1$. This consumption/saving pattern is equivalent to:

$$W(w+y) = -\bar{u} + (1+\bar{u})\cdot(w+y) + \beta V(0) \tag{A16}$$

if $w+y \leq 1$; and

$$W(w+y) = w+y-1+\beta V(1) \tag{A17}$$

if $w+y \geq 2$.

Further, (22) implies: $U(c) - U(1) = c-1 \leq (c-1)\cdot\beta(V(1)-V(0))$ if $c \geq 1$; and $U(c)-U(1) = -(1-c)(1+\bar{u}) \leq (c-1)\cdot\beta(V(1)-V(0))$ if $c \leq 1$. For all $w+y \in [1,2]$, using (23), we have: $W(w+y)) - W(1) = \max_c\{U(c)-U(1)+\beta(V((w+y-c)-V(0)\} \leq \max_c\{(c-1)\cdot\beta(V(1)-V(0))+(w+y-c)\cdot\beta(V(1)-V(0))\} = (w+y-1)\cdot\beta(V(1)-V(0)) = (w+y-1)\cdot(W(2)-W(1))$ or equivalently,

$$W(w+y) \leq (w+y-1)\cdot W(2) + (2-w-y)\cdot W(1)$$

for all $w+y \in [1,2]$. Then, the bargained sum of wealth and income, $w+y$, may probabilistically take on the value of 1 or 2.

Further, (A16) and (A17) imply:

$$W(w+y) - W(w) = y(1+\bar{u}) \geq y\cdot\beta(V(1)-V(0)) = y(W(2)-W(1))$$

for all $w \in [0,1]$ and $y \in [0,1-w]$; and

$$W(w+y) - W(2) = w+y-2 \leq (w+y-2)\cdot\beta(V(1)-V(0)) = (w+y-2)(W(2)-W(1))$$

for all $w \in [0,1]$ and $y \in [2-w,3]$. Then, possible output shares can be represented as a convex set with linear frontiers in the first quadrant of a two dimensional space with the axes of $\sum_y \pi(y;w,1)$ and $\sum_y \pi(y;1,w)$.

Redrawing the frontiers of possible output shares as w changes, we can conjecture the following pattern of the bargaining outcome: there are $\tilde{w}_1$, $\tilde{w}_2$, and $\tilde{w}_3$ with $0 < \tilde{w}_1 < \tilde{w}_2 < \tilde{w}_3 < 1$ such that $\pi(\tilde{y}_1(w); w, 1) = 1$ if $0 \le w \le \tilde{w}_1$; $\pi(\tilde{y}_1(w); w, 1) + \pi(\tilde{y}_2(w); w, 1) = 1$ if $\tilde{w}_1 \le w \le \tilde{w}_2$; $\pi(\tilde{y}_2(w); w, 1) = 1$ if $\tilde{w}_2 \le w \le \tilde{w}_3$; $\pi(\tilde{y}_3(w); w, 1) = 1$ if $\tilde{w}_3 \le w \le 1$, where $\tilde{y}_1(w)$, $\tilde{y}_2(w)$, and $\tilde{y}_3(w)$ solve: $w + \tilde{y}_1(w) = 1$; $w + \tilde{y}_2(w) = 2$;

$$w + \tilde{y}_3(w) = 2 + \frac{w - \tilde{w}_3}{2(1 - \tilde{w}_3)}; \tag{A18}$$

and

$$\pi(\tilde{y}_2(w); w, 1) = \frac{w - \tilde{w}_1}{\tilde{w}_2 - \tilde{w}_1}.$$

Given the assumed consumption/saving behavior, we have: $V(1) = 3/2 + \beta V(1) = 3/(2(1-\beta))$; $V(w) = w + \tilde{y}_3(w) - 1 + \beta V(1)$ if $\tilde{w}_3 \le w \le 1$; $V(w) = 1 + \beta V(1) = (2+\beta)/(2(1-\beta))$ if $\tilde{w}_2 \le w \le \tilde{w}_3$;

$$V(w) = 1 + \beta\big(\pi(\tilde{y}_2(w); w, 1) \cdot V(1) + (1 - \pi(\tilde{y}_2(w); w, 1)) \cdot V(0)\big)$$

if $\tilde{w}_1 \le w \le \tilde{w}_2$; and $V(w) = 1 + \beta V(0) = 1/(1-\beta)$ if $0 \le w \le \tilde{w}_1$.

From the above expressions of $V(w)$, observe that $\beta(V(1) - V(0)) = \beta/(2(1-\beta))$. Then, (22) is equivalent to:

$$1 \le \frac{\beta}{2(1-\beta)} \le 1 + \bar{u} \tag{A19}$$

From the above expressions of $V(w)$, (23) is equivalent to:

$$\beta(V(\tilde{w}_2) - V(0)) \le \tilde{w}_2 \cdot \beta(V(1) - V(0))$$

or to:

$$\tilde{w}_2 \ge \frac{\beta(V(\tilde{w}_2) - V(0))}{\beta(V(1) - V(0))} = \beta. \tag{A20}$$

Now, I will derive the expressions of $\tilde{w}_1$, $\tilde{w}_2$, and $\tilde{w}_3$. If $\tilde{w}_3 \leq w \leq 1$, the first order condition of maximizing the Nash product is:

$$W(4 - \tilde{y}_3(w)) - W(1) - (W(w + \tilde{y}_3(w)) - W(w)) = 0.$$

Using (A17), we have:

$$\tilde{y}_3(w) = \frac{3}{2} - \frac{\bar{u}(1-w)}{2}. \tag{A21}$$

Comparing (A18) and (A21), we have:

$$\tilde{w}_3 = \frac{1+\bar{u}}{2+\bar{u}}.$$

If $\tilde{w}_1 \leq w \leq \tilde{w}_2$, the first order condition of maximizing the Nash product is:

$$(W(2)-W(1))\cdot(\pi(2-w; w,1)\cdot W(2+w)+(1-\pi(2-w; w,1))\cdot W(3+w)-W(1))$$
$$+(W(2+w)-W(3+w))\cdot(\pi(2-w; w,1)\cdot W(2)+(1-\pi(2-w; w,1))\cdot W(1)-W(w)) = 0.$$

Since $\pi(2 - \tilde{w}_2; \tilde{w}_2, 1) = 1$, we have:

$$\beta(V(1) - V(0)) \cdot (\tilde{w}_2 + \beta(V(1) - V(0))) = (1 + \bar{u})(1 - \tilde{w}_2) + \beta(V(1) - V(0))$$

or

$$\tilde{w}_2 = 1 - \frac{\beta^2(V(1) - V(0))^2}{1 + \bar{u} + \beta(V(1) - V(0))}. \tag{A22}$$

Similarly, since $\pi(2 - \tilde{w}_1; \tilde{w}_1, 1) = 0$, we have:

$$\beta(V(1) - V(0)) \cdot (1 + \tilde{w}_1 + \beta(V(1) - V(0))) = (1 + \bar{u})(1 - \tilde{w}_1)$$

or

$$\tilde{w}_1 = \frac{1 + \bar{u} - \beta(V(1) - V(0)) - \beta^2(V(1) - V(0))^2}{1 + \bar{u} + \beta(V(1) - V(0))}.$$

Given (A22), (A20) becomes (24). We can verify that $0 < \tilde{w}_1 < \tilde{w}_2 < \tilde{w}_3 < 1$ given (A19) and (24). Finally, observe that the second inequality in (A19) holds if (24) holds and the first inequality is equivalent to $(17)'$. In summary, we have Proposition 4.

Proof of Proposition 5

Equations (25), (26), and (27) are consistent with the following saving/consumption pattern: $W(0) = -\bar{u} + \beta V(0)$; $W(w + y) = 1 + \beta V(w + y - 1)$ if $w + y \le \bar{w} + 1$; and $W(w + y) = w + y - \bar{w} + \beta V(\bar{w})$ if $w + y \ge \bar{w} + 1$.

For $(w, w') = (\bar{w}, \bar{w}+n)$ with $n \ge 1$, I can show that, holding $\pi(0; \bar{w}, \bar{w}+n)+\pi(1; \bar{w}, \bar{w}+n) = 1$, the first order derivative of the Nash Product in $(4)''$ with respect to $\pi(1; \bar{w}, \bar{w} + n)$ is positive for all values of $\pi(1; \bar{w}, \bar{w}+n)$. Therefore, $\pi(0; \bar{w}, \bar{w}+n) = 0$. Then, the first order condition is $\pi(1; \bar{w}, \bar{w} + n) + \pi(2; \bar{w}, \bar{w} + n) = 1$ and

$$\begin{aligned}&(W(\bar{w}+2)-W(\bar{w}+1))(\breve{\pi}(\bar{w}+n)W(\bar{w}+n+1)+(1-\breve{\pi}(\bar{w}+n))W(\bar{w}+n+2)-W(\bar{w}+n))\\&+(W(\bar{w}+n+1)-W(\bar{w}+n+2))(\breve{\pi}(\bar{w}+n)W(\bar{w}+2)+(1-\breve{\pi}(\bar{w}+n))W(\bar{w}+1)-W(\bar{w}))\le 0\end{aligned} \tag{A23}$$

with the strict equality if $\breve{\pi}(\bar{w} + n) \equiv \pi(2; \bar{w}, \bar{w} + n) > 0$.

For $(w, w') = (\bar{w}, \bar{w}-n)$ with $n \ge 1$, I can show that, holding $\pi(2; \bar{w}, \bar{w}-n)+\pi(3; \bar{w}, \bar{w}-n) = 1$, the first order derivative of the Nash Product in $(4)''$ with respect to $\pi(3; \bar{w}, \bar{w} - n)$ is negative for all values of $\pi(3; \bar{w}, \bar{w} - n)$. Therefore, $\pi(3; \bar{w}, \bar{w} - n) = 0$. Then, the first order condition is $\pi(1; \bar{w}, \bar{w} - n) + \pi(2; \bar{w}, \bar{w} - n) = 1$ and

$$\begin{aligned}&(W(\bar{w}+2)-W(\bar{w}+1))(\breve{\pi}(\bar{w}-n)W(\bar{w}-n+1)+(1-\breve{\pi}(\bar{w}-n))W(\bar{w}-n+2)-W(\bar{w}-n))\\&+(W(\bar{w}-n+1)-W(\bar{w}-n+2))(\breve{\pi}(\bar{w}-n)W(\bar{w}+2)+(1-\breve{\pi}(\bar{w}-n))W(\bar{w}+1)-W(\bar{w}))\ge 0\end{aligned} \tag{A24}$$

with the strict equality if $\breve{\pi}(\bar{w} - n) \equiv \pi(2; \bar{w}, \bar{w} - n) < 1$.

Given the expressions of $W(w)$, (A23) is equivalent to:

$$\breve{\pi}(\bar{w} + n) \ge 1 - \frac{\beta(V(\bar{w}) - V(\bar{w} - 1))}{2} \tag{A25}$$

for $n \ge 1$. Observe that the right-hand sides of (A25) is free of n. Then, $\breve{\pi}(\bar{w}+n) = \breve{\pi}(\bar{w}+m)$ and $\hat{\pi}(\bar{w}+n) = \hat{\pi}(\bar{w}+m)$ for all $n, m \ge 1$. Then, $\beta(V(\bar{w}+n) - V(\bar{w}+n-1)) = \beta < 1$ for all $n \ge 2$. This verifies (27) for $n \ge 2$.

Given the expressions of $W(w)$, (A24) is equivalent to:

$$\breve{\pi}(0) \leq \frac{1}{2}\left(1 - \beta(V(\bar{w}) - V(\bar{w}-1)) + \frac{1+\bar{u}}{\beta(V(1) - V(0))}\right) \tag{29}$$

and

$$\breve{\pi}(\bar{w}-n) \leq \frac{1}{2}\left(1 - \beta(V(\bar{w}) - V(\bar{w}-1)) + \frac{\beta(V(\bar{w}-n) - V(\bar{w}-n-1))}{\beta(V(\bar{w}-n+1) - V(\bar{w}-n))}\right) \tag{A26}$$

for $n \in \{1, 2, \ldots, \bar{w}-1\}$.

Now, the value functions can be written as:

$$V(\bar{w}-n) = 1 + \beta(\breve{\pi}(\bar{w}-n) \cdot V(\bar{w}-n) + (1 - \breve{\pi}(\bar{w}-n)) \cdot V(\bar{w}-n+1)); \tag{A27}$$

$$V(\bar{w}) = \frac{3}{2} + \beta V(\bar{w}); \tag{A28}$$

and

$$V(\bar{w}+n) = \breve{\pi}(\bar{w}+1) \cdot (n+1) + (1 - \breve{\pi}(\bar{w}+1)) \cdot (n+2) + \beta V(\bar{w}), \tag{A29}$$

where $n \geq 1$.

In order for the first inequality in (25) to hold for $n = 0$, in (A27) and (A28), we need:

$$\beta(V(\bar{w}) - V(\bar{w}-1)) = \frac{\beta}{2(1 - \beta\breve{\pi}(\bar{w}-1))} \geq 1 \tag{A30}$$

or

$$\breve{\pi}(\bar{w}-1) \geq \pi_0 \equiv \frac{1}{\beta} - \frac{1}{2}. \tag*{(16)$'$}$$

Since $\breve{\pi}(\bar{w}-1) \leq 1$, a necessary condition for the equilibrium with $\bar{w} \geq 1$ is:

$$\beta \geq \frac{2}{3}. \tag*{(17)$'$}$$

In order for (27) to hold for $n = 1$, in (A28) and (A29), we need:

$$\beta(V(\bar{w}+1) - V(\bar{w})) = \beta\left(\frac{3}{2} - \breve{\pi}(\bar{w}+1)\right) \leq 1 \tag{A31}$$

or

$$\breve{\pi}(\bar{w}+1) \geq \frac{3}{2} - \frac{1}{\beta}. \tag{A32}$$

From (A25) and (A30), $\breve{\pi}(\bar{w}+1) \geq 1 - \beta/(4(1-\beta))$. Combined with (A32), this implies that $\breve{\pi}(\bar{w}+1) > 0$ given (17). Then, (A25) holds with equality. Then, from (A32), (A30), and (A25), we have:

$$\breve{\pi}(\bar{w}-1) \leq \pi_1 \equiv \min\left\{1, \frac{1}{\beta} - \frac{1}{2} \cdot \frac{\beta}{2-\beta}\right\}. \tag{A33}$$

The equilibrium value of $\breve{\pi}(\bar{w}-1)$ is limited to the range $[\pi_0, \pi_1]$, given (16) and (A33).

In order for (26) to hold, in (A27) and (A28), further we need:

$$\frac{\beta(V(\bar{w}-n) - V(\bar{w}-n-1))}{\beta(V(\bar{w}-n+1) - V(\bar{w}-n))} = \frac{\beta(1-\breve{\pi}(\bar{w}-n))}{1-\beta\breve{\pi}(\bar{w}-n-1)} \geq 1 \tag{A34}$$

or

$$\breve{\pi}(\bar{w}-n-1) - \breve{\pi}(\bar{w}-n) \geq \frac{1}{\beta} - 1 \tag{A35}$$

for all $n \in \{1, 2, \ldots, \bar{w}-1\}$.

Note that (A35) implies that $\breve{\pi}(\bar{w}-n) < 1$ for all $n \in \{1, 2, \ldots, \bar{w}-1\}$. Then, (A26) holds with equality. Then, from (A30), (A34), and (A26), we have:

$$\frac{\beta(1-\breve{\pi}(\bar{w}-n))}{1-\beta\breve{\pi}(\bar{w}-n-1)} = 2\breve{\pi}(\bar{w}-n) - 1 + \frac{\beta}{2(1-\beta\breve{\pi}(\bar{w}-1))}, \tag{28}$$

which yields $\breve{\pi}(\bar{w}-n-1)$ given $\breve{\pi}(\bar{w}-n)$.

Assume (17) and consider the following algorithm of finding the equilibrium values. Pick a value of $\breve{\pi}(\bar{w}-1)$ between π_0 and π_1 in (16) and (A33). Then, (A31) and (A30) hold by construction. Derive $\{\breve{\pi}(\bar{w}-n)\}$ for $n \geq 2$ from (28). From (A30) and (16), observe that the righthand side of (28) is greater than 1. Then, (A35) holds. Given (A35), observe that there is $\bar{n} \geq 1$ so that $\breve{\pi}(\bar{w}-\bar{n}) \leq 1$ and there is no $\breve{\pi}(\bar{w}-\bar{n}-1) \leq 1$ that solves (28). Then, the maximum possible equilibrium value of $\bar{w}$ is $\bar{n}$. For any $\bar{w} \leq \bar{n}$, derive $\bar{u}$ that solves (29) with equality given (A30) and (A34). Given (16), (17), and (A35), $\breve{\pi}(0) \geq \pi_0 > 1/2$. Given (A30), $\beta(V(\bar{w}) - V(\bar{w}-1) > 1$. Then, $1+\bar{u} > \beta(V(1) - V(0))$ in (29). This verifies the second inequality in (25). Finally, set $\breve{\pi}(\bar{w}) = 1/2$; derive $\breve{\pi}(\bar{w}+1)$ from (A25) and (A30); and set $\breve{\pi}(\bar{w}+n) = \breve{\pi}(\bar{w}+1)$ for all $n \geq 2$. Then, by construction, $\bar{w}$, $\bar{u}$, and $\{\breve{\pi}(n)\}_{n\geq 0}$ are an equilibrium.

The above algorithm shows that any $\bar{w} \in \{1, 2, \ldots, \bar{n}\}$, where $\bar{n}$ is determined by $\breve{\pi}(\bar{w}-1)$, is supported as an equilibrium by some value of $\bar{u}$. In order to keep track of the multiple equilibria, let $\bar{n}|\breve{\pi}(\bar{w}-1)$ denote $\bar{n}$ given $\breve{\pi}(\bar{w}-1)$. Let $\breve{\pi}(w)|(\bar{w}, \breve{\pi}(\bar{w}-1))$ denote $\breve{\pi}(w)$ given $\bar{w}$ and $\breve{\pi}(\bar{w}-1)$. Similarly, let $\bar{u}|(\bar{w}, \breve{\pi}(\bar{w}-1))$ denote $\bar{u}$ that solves (29) with equality, given $\bar{w}$ and $\breve{\pi}(\bar{w}-1)$. Let $V(w)|(\bar{w}, \breve{\pi}(\bar{w}-1))$ denote $V(w)$ given $\bar{w}$ and $\breve{\pi}(\bar{w}-1)$.

Observe that $\breve{\pi}(w+1)|(\bar{w}+1, \breve{\pi}(\bar{w}-1)) = \breve{\pi}(w)|(\bar{w}, \breve{\pi}(\bar{w}-1)) > \breve{\pi}(w+1)|(\bar{w}, \breve{\pi}(\bar{w}-1))$ if $w, w+1 \in \{1, \bar{n}|\breve{\pi}(\bar{w}-1)\}$. In particular, $\breve{\pi}(0)|(\bar{w}+1, \breve{\pi}(\bar{w}-1)) > \breve{\pi}(0)|(\bar{w}, \breve{\pi}(\bar{w}-1))$. Similarly, we can show that $V(1)|(\bar{w}+1, \breve{\pi}(\bar{w}-1)) - V(0)|(\bar{w}+1, \breve{\pi}(\bar{w}-1)) > V(1)|(\bar{w}, \breve{\pi}(\bar{w}-1)) - V(0)|(\bar{w}, \breve{\pi}(\bar{w}-1))$. Then, $\bar{u}|(\bar{w}+1, \breve{\pi}(\bar{w}-1)) > \bar{u}|(\bar{w}, \breve{\pi}(\bar{w}-1))$ in (29) if $w, w+1 \in \{1, \bar{n}|\breve{\pi}(\bar{w}-1)\}$.

Observe that $\breve{\pi}(\bar{w}-n-1)$ as a function of $\breve{\pi}(\bar{w}-n)$ is continuous and increasing in (28) so that $\breve{\pi}(w)|(\bar{w}, \breve{\pi}(\bar{w}-1))$ is continuous and increasing in $\breve{\pi}(\bar{w}-1)$, holding w and $\bar{w}$, while $\breve{\pi}(\bar{w}-1) \in [\pi_0, \pi_1]$. Relatedly, the lefthand side of (28) or, equivalently, the righthand side of (A34) as a function of $\breve{\pi}(\bar{w}-1)$ is continuous and increasing. Consulting (A30), observe that $V(1)|(\bar{w}, \breve{\pi}(\bar{w}-1)) - V(0)|(\bar{w}, \breve{\pi}(\bar{w}-1))$ is continuous and increasing in $\breve{\pi}(\bar{w}-1)$. Putting these properties together, observe that $\bar{u}|(\bar{w}, \breve{\pi}(\bar{w}-1))$ is continuous and increasing in $\breve{\pi}(\bar{w}-1)$ in (29). Then, if $\bar{w} \leq \bar{n}|\pi_1$ and $\bar{u}|(\bar{w}, \pi_0) < \bar{u} < \bar{u}|(\bar{w}, \pi_1)$ for some $\bar{u}$, there is $x \in (\pi_0, \pi_1)$ so that $\bar{w} \leq \bar{n}|x$ and $\bar{u}|x = \bar{u}$. Therefore, for any $\bar{w} \leq \bar{n}|\pi_1$

and $\bar{u} \in [\bar{u}|(\bar{w}, \pi_0), \bar{u}|(\bar{w}, \pi_1)]$, there is $\check{\pi}(\bar{w}-1)$ so that they along with $\{\check{\pi}(w)\}$ derived by the above algorithm is an equilibrium.

Now suppose that $\check{\pi}(0)|(\bar{n}|\pi_1, \pi_1) = 1$. Then, in (29), observe that any $\bar{u} \geq \bar{u}|(\bar{n}|\pi_1, \pi_1)$ along with the other equilibrium values associated with $\check{\pi}(\bar{w}-1) = \pi_1$ is an equilibrium by construction. Therefore, the range of $\bar{u}$ that supports the equilibrium with $\bar{w} = \bar{n}|\pi_1$ is extended to the infinite set $[\bar{u}|(\bar{n}, \pi_0), \infty)$ if $\check{\pi}(0)|(\bar{n}|\pi_1, \pi_1) = 1$. Note that, for $\bar{w} < \bar{n}|\pi_1$, the range of $\bar{u}$ that supports the equilibrium with $\bar{w}$ is necessarily the finite set $[\bar{u}|(\bar{w}, \pi_0), \bar{u}|(\bar{w}, \pi_1)]$ since $\bar{u}|(\bar{w}, \pi_1)$ is increasing in $\bar{w}$.

Now suppose that $\bar{n}|\pi_1 < \bar{n}|\pi_0$. Then, for any $\bar{w} \in \{\bar{n}|\pi_1 + 1, \bar{n}|\pi_0\}$, $\check{\pi}(0)|(\bar{w}, \pi_0)$ and $\bar{u}|(\bar{w}, \pi_0)$ exist while $\check{\pi}(0)|(\bar{w}, \pi_1)$ and $\bar{u}|(\bar{w}, \pi_1)$ do not exist. Since $\bar{u}|(\bar{w}, \check{\pi}(\bar{w}-1))$ is continuous and increasing in $\check{\pi}(\bar{w}-1)$, there is $\check{\pi}(\bar{w}-1) \in [\pi_0, \pi_1]$, call it π_2, so that $\check{\pi}(0)|(\bar{w}, \pi_2) = 1$. Then, any $\bar{u} \in [\bar{u}|(\bar{w}, \pi_0), \bar{u}|(\bar{w}, \pi_2)]$ is an equilibrium for some $\check{\pi}(\bar{w}-1) \in [\pi_0, \pi_2]$ by the above algorithm. Further, any $\bar{u} \geq \bar{u}|(\bar{w}, \pi_2)$ is an equilibrium along with the other equilibrium values associated with $\check{\pi}(\bar{w}-1) = \pi_2$, repeating the reasoning in the above. Therefore, for any $\bar{w} \in \{\bar{n}|\pi_1 + 1, \bar{n}|\pi_0\}$, the range of $\bar{u}$ that supports the equilibrium with $\bar{w}$ is necessarily the infinite set $[\bar{u}|(\bar{w}, \pi_0), \infty)$.

In the remainder of the proof, we will show that the union of the ranges of $\bar{u}$ that support the equilibria with various values of $\bar{w}$ is contiguous, i.e., if $x_1 < x_2$ and if there are an equilibrium with $\bar{u} = x_1$ and another equilibrium with $\bar{u} = x_2$, there is an equilibrium with $\bar{u} = x$ for any $x \in (x_1, x_2)$. The ranges of $\bar{u}$ that support the equilibria with $\bar{w} \in \{\bar{n}|\pi_1 + 1, \bar{n}|\pi_0\}$, if any, are the infinite sets, so the union of these sets is contiguous. If $\check{\pi}(0)|(\bar{n}|\pi_1, \pi_1) = 1$, the range of $\bar{u}$ that supports the equilibrium with $\bar{w} = \bar{n}|\pi_1$ is also infinite so that the union of the ranges of $\bar{u}$ that support the equilibria with $\bar{w} \in \{\bar{n}|\pi_1, \bar{n}|\pi_0\}$ is contiguous. Therefore, the union of the ranges of $\bar{u}$ that support any equilibria is contiguous if $\bar{u}|(\bar{w},\pi_0) < \bar{u}|(\bar{w}-1,\pi_1)$ for any $\bar{w}$ with $\bar{w} \le \bar{n}|\pi_0$, $\bar{w}-1 \le \bar{n}|\pi_1$, and $\breve{\pi}(0)|(\bar{w}-1,\pi_1) < 1$.

If $\pi_1 = 1$, $\bar{n}|\pi_1 = 1$ and $\breve{\pi}(0)|(\bar{n}|\pi_1,\pi_1) = \pi_1 = 1$ so that the union of the ranges of $\bar{u}$ that support any equilibria is contiguous. Consider the case of $\pi_1 < 1$ in the following. From (A34) and (28), we have:

$$\frac{\beta(V(\bar{w}-n)-V(\bar{w}-n-1))}{\beta(V(\bar{w}-n+1)-V(\bar{w}-n))} = 2\breve{\pi}(\bar{w}-n) - 1 + \frac{\beta}{2(1-\beta\breve{\pi}(\bar{w}-1))}. \tag{A36}$$

Equation (A36) holds for $n \ge 1$; it also holds for $n = 0$ if $\breve{\pi}(\bar{w}-1) = \pi_1$ since $\breve{\pi}(\bar{w}) = 1/2$ and $\beta(V(\bar{w}+1)-V(\bar{w})) = 1$ if $\breve{\pi}(\bar{w}-1) = \pi_1$, given (A31), (A32), and (A33). From (A30) and (16), $\beta/(2(1-\beta\pi_0)) = 1$. From (A25), (A32), (A30), (16), and (A33), $\beta/(2(1-\beta\pi_1)) = 2\pi_0$. Then, from (A36), we have:

$$\frac{\Delta(w-1)|(\bar{w}-1,\pi_1)}{\Delta(w-1)|(\bar{w},\pi_0)} = \frac{2\breve{\pi}(w)|(\bar{w}-1,\pi_1) + 2\pi_0 - 1}{2\breve{\pi}(w)|(\bar{w},\pi_0)} \cdot \frac{\Delta(w)|(\bar{w}-1,\pi_1)}{\Delta(w)|(\bar{w},\pi_0)}, \tag{A37}$$

where $w \le \bar{w}-1$, $\Delta(w)|(\bar{w},\breve{\pi}(\bar{w}-1)) \equiv \beta(V(w+1)|(\bar{w},\breve{\pi}(\bar{w}-1)) - V(w)|(\bar{w},\breve{\pi}(\bar{w}-1)))$, and $V(w)|(\bar{w},\breve{\pi}(\bar{w}-1))$ is $V(w)$ given $\bar{w}$ and $\breve{\pi}(\bar{w}-1)$. From (28), we can derive:

$$2\breve{\pi}(w-1)|(\bar{w},\pi_0) = \frac{2}{\beta} + 1 - \frac{2}{2\breve{\pi}(w)|(\bar{w},\pi_0)} \tag{A38}$$

and

$$2\breve{\pi}(w-1)|(\bar{w},\pi_1) + 2\pi_0 - 1 = \frac{2}{\beta} + 2\pi_0 - \frac{2\pi_0+1}{2\breve{\pi}(w)|(\bar{w},\pi_1) + 2\pi_0 - 1}. \tag{A39}$$

Note that (A38) and (A39) define two functions, F and G: $2\breve{\pi}(w-1)|(\bar{w},\pi_0) = F(2\breve{\pi}(w)|(\bar{w},\pi_0))$ and $2\breve{\pi}(w-1)|(\bar{w},\pi_1) + 2\pi_0 - 1 = G(2\breve{\pi}(w)|(\bar{w},\pi_1) + 2\pi_0 - 1)$. Further, note that $F(z) < G(z)$ for all $z \ge 1$. Let $F^{t+1}(z) \equiv F(F^t(z))$ and $G^{t+1}(z) \equiv G(G^t(z))$ for all $t \ge 1$. Then, from (A37), we have:

$$\begin{aligned}\frac{\Delta(w)|(\bar{w}-1,\pi_1)}{\Delta(w)|(\bar{w},\pi_0)} &= \frac{G^{\bar{w}-w-1}(2\breve{\pi}(\bar{w}-1)|(\bar{w}-1,\pi_1)+2\pi_0-1)}{F^{\bar{w}-w-1}(2\breve{\pi}(\bar{w}-1)|(\bar{w},\pi_0))} \cdot \frac{\Delta(\bar{w}-1)|(\bar{w}-1,\pi_1)}{\Delta(\bar{w}-1)|(\bar{w},\pi_0)} \\ &= \frac{G^{\bar{w}-w-1}(2\pi_0)}{F^{\bar{w}-w-1}(2\pi_0)} \\ &> 1,\end{aligned} \tag{A40}$$

where the second equality uses: $\check{\pi}(\bar{w}-1)|(\bar{w}-1,\pi_1)=1/2$; $\check{\pi}(\bar{w}-1)|(\bar{w},\pi_0)=\pi_0$; $\Delta(\bar{w}-1)|(\bar{w}-1,\pi_1)=\beta(V(\bar{w})|(\bar{w}-1,\pi_1)-V(\bar{w}-1)|(\bar{w}-1,\pi_1))=1$; and $\Delta(\bar{w}-1)|(\bar{w},\pi_0)=\beta(V(\bar{w})|(\bar{w},\pi_0)-V(\bar{w}-1)|(\bar{w},\pi_0))=1$. From (29), (A37), and (A40), we have:

$$\frac{1+\bar{u}|(\bar{w}-1,\pi_1)}{1+\bar{u}|(\bar{w},\pi_0)}=\frac{2\check{\pi}(0)|(\bar{w}-1,\pi_1)+2\pi_0-1}{2\check{\pi}(0)|(\bar{w},\pi_0)}\cdot\frac{\Delta(0|(\bar{w}-1,\pi_1))}{\Delta(0|(\bar{w},\pi_0))}=\frac{G^{\bar{w}}(2\pi_0)}{F^{\bar{w}}(2\pi_0)}>1$$

if $\bar{w}\leq\bar{n}|\pi_0$ and $\bar{w}-1\leq\bar{n}|\pi_1$. Therefore, the union of the ranges of $\bar{u}$ that support any equilibria is contiguous.

In order to summarize, let $\bar{\omega}\equiv\bar{n}|\pi_0$; $\check{u}(\bar{w})\equiv\bar{u}|(\bar{w},\pi_0)$; and $\hat{u}(\bar{w})\equiv\bar{u}|(\bar{w},\pi_1)$. We then have Proposition 5.